# Marking game and graph operators

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**Abstract**

The marking game is a two players game played on a graph, associated to the *game coloring number* of the graph. It was first described as a colorblind version of the coloring game, which was the reason why this game was studied at first.

We here study how the game coloring number changes on basic operations on the graph. We are interested mainly about the deletion of a vertex or an edge, about the contraction of edges, the union of two graphs and the Cartesian product. Moreover, we show that our bounds are tight and give examples of infinite families of graphs attaining these bounds, that we introduce as the *class of sunflowers*.

The *graph marking game* is a game on a graph that was first introduced in 1998 by Zhu [3] as a tool to study the *graph coloring game*, introduced in 1981 by Brams in [1] and rediscovered ten years later by Bodlaender, [2]. The graph marking game is played by two players, Alice and Bob, who alternate turns marking a yet unmarked vertex, until the whole graph is marked. Each time a vertex is marked, its score is 1 plus the number of its marked neighbors. At the end, the score of the graph is the maximum score of a vertex during the game. Alice tries to minimize it while Bob tries to maximize it. When Alice starts and both players play optimally, the score of the game is then known as the *game coloring number* of the graph $G$ and is denoted by $col_g(G)$.

In this paper we study in details how the game coloring number of a graph may change when small modifications are made to the graph. Among earlier results on this topic, Zhu proved in [3] the following result about subgraphs.

**Theorem 1** (Zhu [3])**.** *For a graph $G$, let $G_1$ and $G_2$ be spanning subgraphs of $G$ such that $E(G) = E(G_1) \cup E(G_2)$ and $E(G_1) \cap E(G_2) = \varnothing$. We have*

$$col_g(G) \leq col_g(G_1) + \Delta(G_2)$$

*where $\Delta(G_2)$ denotes the maximum degree of $G_2$.*

Note that this implies in particular that the game coloring number of a spanning subgraph is at most the game coloring number of $G$. Inspired by this, Sia proved the following about the Cartesian product of two graphs:

**Theorem 2** (Sia [4])**.** *Let $G$ and $H$ be two graphs and denote by $\square$ the Cartesian product and $\sqcup$ the disjoint union. Then*

$$col_g(G \square H) \leq col_g\left(\bigsqcup_{h \in V(H)} \{(g,h) \mid g \in G\}\right) + \Delta(H).$$

Remark that the set of vertices of $C = G\square H$ is $V(C) = \{(g,h) \mid g \in G, h \in H\}$.

For the following study, we need some notations on the game and some variants. First, we consider the marking game and a variant where Bob has the first move. To avoid any confusion, we call *A-marking game* the game for which Alice has the first move, and *B-marking game* the game where Bob has the first move. We denote $col_A(G)$ and $col_B(G)$ the scores of these games respectively when both players play optimally, called $A$-coloring number and $B$-coloring number (recall that the default game is the $A$-marking game). When the variant needs not be explicit, we use the notation $col_X$, $X \in \{A, B\}$. We also consider games under progress. For a subset $M \subseteq V(G)$, denote $G|M$ the graph $G$ where the vertices of $M$ have been marked already. When considering the score of the games, the score of the vertices of $M$ is not considered though they are counted as marked neighbors for the score of vertices in $V \setminus M$.

We say that *Alice has a strategy for score $s$* in a game if she can ensure that no vertex get a score larger than $s$. In particular, $col_X(G|M)$ is then the smallest positive number $k$ such that Alice has a strategy for score $k$ in the $X$-marking game on $G|M$. Similarly, we say that Bob has a strategy for score $s$ on a graph $G$ for the $X$-marking game if he has a strategy such that, no matter how Alice plays, there is always a vertex with score at least $s$. Now, $s = col_X(G|M)$ is the only value for which both Alice and Bob have a strategy for score $s$.

Our first result states that starting the game is an advantage in any situation and quantifies this advantage.

**Theorem 3.** *Let $G$ be a graph with a set $M$ of marked vertices. Then:*

$$col_A(G|M) \leq col_B(G|M) \leq col_A(G|M) + 1.$$

*Sketch of the proof.* Let us prove $col_B(G|M) \leq col_A(G[M) + 1$. Alice has a strategy for score $s$ for the $A$-marking game. Playing the $B$-game, she just ignores Bob's first move and she plays her strategy. This way vertices have score at most $s+1$, hence $col_B(G|M) \leq col_A(G|M)+1$. Similarly, if she has a strategy for score $s$ for the $B$-marking game, then, when playing to the $A$-one, she imagines the first move of Bob and she plays accordingly. By doing this, in the $A$-marking game vertices have at most the same score than in the $B$-marking game, thus $col_A(G|M) \leq col_B(G|M)$. □

In particular, this result implies the following corollary:

**Corollary 1.** *If Alice has a strategy for score $s$ on the graph $G|M$, then if Bob passes a turn, she still has a strategy for score $s$. If Bob has a strategy for score $s$ on a graph $G|M$, then if Alice passes a turn, he still has a strategy for score $s$.*

In the following, most examples of tightness are based on *sunflower* graphs, which are formed by a clique and a stable set *joined* in a particular regular way.

**Definition 1.** *Let $n$ and $k$ be positive integers such that $k \leq n$. We define the sunflower $SF_{n,k}$ with:*

- $V(SF_{n,k}) = \{a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}\}$
- $E(SF_{n,k}) = \{a_i a_j | 0 \leq i < j < n, i \neq j\} \cup \{a_i b_j | 0 \leq i < n, j \in \{i, (i+1) \bmod n, \ldots, (i+k-1) \bmod n\}\}$.

*We call* seed vertices *those from the set $A = \{a_0, \ldots, a_{n-1}\}$, and* petal vertices *those from the set $B = \{b_0, \ldots, b_{n-1}\}$. We define also the graph $SF^*_{n,k}$ as the graph $SF_{n,k} - \{b_0\}$.*




Figure 1: Example of sunflower $\mathrm{SF}_{6,2}$ and $\mathrm{SF}^*_{6,3}$

Examples of sunflowers $\mathrm{SF}_{n,k}$ and $\mathrm{SF}^*_{n,k}$ are drawn in Figure 1.

For vertex deletion, we denote by $G-\{v\}$ the graph $(V(G)\setminus\{v\}, E(G)\setminus\{(u,v) \mid u \in V(G), u \neq v\})$, obtained from $G$ by deleting the vertex $v$ and all incident edges.

**Theorem 4.** *Let $G$ be a graph, $M$ a set of marked vertices of $G$ and $v$ a vertex of $V(G)\setminus M$ with $|N(v)\cap M| < col_A(G)$. If $G-\{v\}$ has at least one unmarked vertex, then*

$$col_A(G|M) - 2 \leq col_A(G-\{v\}|M) \leq col_A(G|M),$$

*and both bounds are tight. Moreover the lower bound is tight only for vertices of degree at least $col_A(G)-1$.*

One family of graphs attaining the lower bound is the family $\mathrm{SF}^*_{n,n}$ for $v = a_0$. For the upper bound we have $\mathrm{SF}_{n,n}$ for $v = b_0$.

For edge deletion, we denote by $G\setminus\{e\}$ the graph $(V(G), E(G)\setminus\{e\})$, obtained by deleting the edge $e$.

**Theorem 5.** *Let $G$ be a graph, $M$ a set of marked vertices of $G$ and $e$ an edge of $G$. We assume $G$ has at least one unmarked vertex. We then have:*

$$col_A(G|M) - 1 \leq col_A(G\setminus\{e\}|M) \leq col_A(G|M),$$

*and these bounds are tight.*

For the lower bound, we have the family of graphs $\mathrm{SF}_{n,n}$ when removing the edge $a_0a_1$. For the upper one we have the graphs $\mathrm{SF}_{n,1}$, $n > 2$, when removing $a_0b_0$.

For edge-contraction, we denote $G/e$ the graph obtained by the contraction of the edge $e$, (where the two endpoints of $e$, say $u$ and $v$ are contracted into a unique vertex $w$).

**Theorem 6.** *Let $G$ be a graph, $e = uv$ an edge of $G$.*

$$col_A(G|M) - 2 \leq col_A(G/e|M) \leq col_A(G|M) + 2,$$

*and both bounds are tight.*

For the upper bound, the tightness is reached for the graphs $\mathrm{SF}^*_{n+1,n+1}$ by contracting the edge $a_0a_1$, hence obtaining the graph $\mathrm{SF}_{n,n}$.
For the lower bound we have the graph obtained by taking for $G$ a subgraph of $\mathrm{SF}^*_{n+1,n+1}$ where we remove some edges in the following way: $a_ib_i$ for $1 \leq i < n+1$, the edges $a_{n+1}b_i$ for $i \geq \lfloor n/2 \rfloor$ and the edges $a_0b_i$ for $i \leq \lfloor n/2 \rfloor$ and for $i = n$. We denote by $d(v)$ the degree of vertex $v$. In this case we have $d(a_0) = d(a_{n+1}) + 1$ if $n$ is even and $d(a_0) = d(a_{n+1})$ otherwise.
For the union of graphs:

**Theorem 7.** *Let $G$ and $H$ be two graphs:*

$$col_A(G \cup H) = \min \left\{ \begin{array}{l} \max(col_A(G), col_B(H)) \\ \max(col_B(G), col_A(H)) \end{array} \right\}.$$

Hence, we can directly improve Sia's result:

**Corollary 2** (Follows from Theorems 2 and 7)**.** *Let $G$ and $H$ be two graphs. Then:*

$$col_A(G \square H) \leq \min \left\{ \begin{array}{l} col_B(G) + \Delta(H) \\ col_B(H) + \Delta(G) \end{array} \right\},$$

*and this bound is tight.*

If $K_\ell$ is the clique on $\ell \geq 2$ vertices, then the graphs $\mathrm{SF}_{n,k} \square K_\ell$, with $n \geq 2$ and $k \geq 2$, reach this upper bound.

# References


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