Eternal domination on digraphs and orientations of graphs

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Abstract

We study the eternal dominating number and the m-eternal dominating number on digraphs. We generalize known results on graphs to digraphs. We also consider the problem "oriented (m-)eternal domination", consisting in finding an orientation of a graph that minimizes its eternal dominating number. We prove that computing the oriented eternal dominating number is NP-hard and characterize the graphs for which its value is 2. We also study these two parameters on trees, cycles, cliques, bicliques and different kinds of grids.

1 Introduction

The problem of eternal domination on undirected graphs, while being a rather recent problem, has been widely studied (see [5] for a survey). It has initially been motivated by problems in military defense.

The eternal domination on a graph G can be seen as an infinite game between two players: the *defender* and the *attacker*. First, the defender chooses a set D_0 of k vertices, with k fixed, called *guards*. At turn i, the attacker chooses a vertex r_i called *attack* in $V \setminus D_{i-1}$ and the defender must *defend* the attack by moving to v_i exactly one guard along one edge, from a vertex v_i adjacent to r_i. The new guards configuration is $D_i = D_{i-1} \cup \{r_i\} \setminus \{v_i\}$. The defender wins the game if he can defend any infinite sequence of attacks. The *eternal domination number*, denoted by $\gamma^\infty(G)$, is the minimum number of guards necessary for the defender to win. An *eternal dominating set* is a set that can initially be chosen by the defender in a winning strategy.

The m-eternal domination is similar to the eternal domination but the defender is authorized to move as many guards as he wants among the k guards at each turn. The m-eternal dominating number is denoted by $\gamma_m^\infty(G)$.

In general graphs, the two following results compare the values of both the eternal domination number and the m-eternal domination number to the values of other well known graph parameters.

Theorem 1. *[1, 3, 4] Given a graph G, we have*

$$\gamma(G) \leq \gamma_m^\infty(G) \leq \alpha(G) \leq \gamma^\infty(G) \leq \binom{\alpha(G)+1}{2}$$

where $\gamma(G)$ is the domination number of G, and $\alpha(G)$ is the independent set number of G.

Theorem 2. *[1] Given a graph G, we have*

$$\gamma^\infty(G) \leq \theta(G)$$

where $\theta(G)$ is the clique covering number of G.

The values of γ^∞ and γ_m^∞ have been studied for many classes of graphs and, in particular, grids.

To our knowledge, the eternal domination problem has only been studied on undirected graphs. In this paper, we consider eternal domination on directed graphs where the guards must follow

the direction of the arcs, both when they defend a vertex and when they move to another vertex. Additionally, as it has been done for many digraph parameters such as diameter, chromatic number, domination number or maximum outgoing degree, we consider the problem consisting in orientating an undirected graph to minimize γ^∞ or γ_m^∞.

2 Eternal domination on digraphs

The main result of this section is the generalization of Theorem 1 to directed graphs. We first define the parameter α for digraphs:

Definition 1. *For G a digraph, $\alpha(G)$ denotes the order of the greatest induced acyclic subgraph of G.*

Notice that this parameter is a generalization of $\alpha(G)$ for graphs since $\alpha(G) = \alpha(\overleftrightarrow{G})$ for any graph G where $\overleftrightarrow{G}$ denotes the symmetrical orientation of G. The analogue of Theorem 1 gives:

Theorem 3. *Given a digraph G, we have*

$$\gamma(G) \leq \gamma_m^\infty(G) \leq \alpha(G) \leq \gamma^\infty(G) \leq \binom{\alpha(G)+1}{2}.$$

3 Eternal domination on orientations of graphs

3.1 Definitions

We now orientate undirected graphs in order to minimize their eternal or m-eternal domination numbers. An orientation of an undirected graph G is an assignment of exactly one direction to each of the edges of G. This leads to the introduction of three new parameters for undirected graphs:

Definition 2. $\overrightarrow{\gamma^\infty}(G) = \min\{\gamma^\infty(H) : H \text{ is an orientation of } G\}$
$\overrightarrow{\gamma_m^\infty}(G) = \min\{\gamma_m^\infty(H) : H \text{ is an orientation of } G\}$
$\overrightarrow{\alpha}(G) = \min\{\alpha(H) : H \text{ is an orientation of } G\}$

3.2 General results

We first give general results for undirected graphs, such as the inequality linking $\overrightarrow{\alpha}(G)$ to $\overrightarrow{\gamma^\infty}(G)$.

Lemma 1. $\overrightarrow{\alpha}(G) \leq \overrightarrow{\gamma^\infty}(G)$.

Let G be an undirected graph. We define $C(G)$ by starting from G, adding one vertex per edge of G and connecting each new vertex to the extremities of the associated edge. This definition allows us to present the following result:

Lemma 2. *Let G be an undirected graph with m edges. Then*

- $\overrightarrow{\gamma^\infty}(C(G)) = \gamma^\infty(G) + m$
- $\overrightarrow{\alpha}(C(G)) = \alpha(G) + m$

The consequences of this lemma are particularly interesting, leading to complexity results:

Corollary 1. *For every $k > 0$, there exists a graph G such that $\overrightarrow{\gamma^\infty}(G) \geq \overrightarrow{\alpha}(G) + k$.*

Corollary 2. *Computing $\overrightarrow{\gamma^\infty}(G)$ is NP-hard.*

Corollary 3. *Computing $\overrightarrow{\alpha}(G)$ is NP-hard.*

3.3 Results on some classes of graphs

We are now interested in the value of $\overrightarrow{\gamma^\infty}$ and $\overrightarrow{\gamma_m^\infty}$ for particular classes of graphs.

The first graphs to be studied are the forests. Since they are acyclic for any orientation, we obtain the following result:

Theorem 4. *Let G be a graph with order n. Then, $\overrightarrow{\gamma^\infty}(G) = n$ iff $\overrightarrow{\gamma_m^\infty}(G) = n$ iff G is a forest.*

The case of cycle is also quite straightforward for both parameters.

Theorem 5. $\overrightarrow{\gamma^\infty}(C_n) = n - 1$ *and* $\overrightarrow{\gamma_m^\infty}(C_n) = \lceil \frac{n}{2} \rceil$ *for every* $n \geq 3$.

Studying the value of $\overrightarrow{\gamma_m^\infty}$ for complete graphs led to a full characterization of graphs G for which $\overrightarrow{\gamma_m^\infty}(G) = 2$.

Theorem 6. *Let G be a graph of order $n \geq 3$. Then, $\overrightarrow{\gamma_m^\infty}(G) = 2$ iff either:*

- *$n = 2k$ and G is a complete graph from which at most k disjoint edges are removed*
- *$n = 2k + 1$ and G is a complete graph from which at most $k - 1$ disjoint edges are removed.*

The exact value of $\overrightarrow{\gamma^\infty}$ for complete graphs seems hard to find and we could only obtain bounds by combining Theorem 3 and a result from Erdős and Moser concerning $\overrightarrow{\alpha}$ [2].

Theorem 7. $\lfloor \log_2 n \rfloor + 1 \leq \overrightarrow{\gamma^\infty}(K_n) \leq \binom{2\lfloor \log_2 n \rfloor + 2}{2}$.

The case of bicliques, on the other hand, has been fully covered for both parameters.

Theorem 8. $\overrightarrow{\gamma^\infty}(K_{n,m}) = \max\{n, m\} + 1$ *for every* $n, m \geq 1$.

Theorem 9. $\overrightarrow{\gamma_m^\infty}(K_{2,2}) = 2$
$\overrightarrow{\gamma_m^\infty}(K_{2,3}) = \overrightarrow{\gamma_m^\infty}(K_{3,3}) = 3$
$\overrightarrow{\gamma_m^\infty}(K_{n,m}) = 4$ *for every* $n \geq 2$ *and* $m \geq 4$.

We have considered $\overrightarrow{\gamma^\infty}$ on square grids and have obtained the following results:

Theorem 10.

$$\left\lceil \frac{2mn}{3} \right\rceil \leq \overrightarrow{\gamma^\infty}(P_n \Box P_m) \leq \left\lceil \frac{7mn + 2m + 2n}{9} \right\rceil.$$

When m and n are both multiples of 3, we have:

$$\overrightarrow{\gamma^\infty}(P_n \Box P_m) \leq \frac{7mn}{9}.$$

For square grids of size $2 \times n$, $3 \times n$ and $4 \times n$, we have the exact value of $\overrightarrow{\gamma^\infty}$.

Theorem 11. *Let $n \geq 2$. Then,* $\overrightarrow{\gamma^\infty}(P_2 \Box P_n) = \lceil \frac{3n}{2} \rceil$,
$\overrightarrow{\gamma^\infty}(P_3 \Box P_n) = \lceil \frac{7n}{3} \rceil$,
$\overrightarrow{\gamma^\infty}(P_4 \Box P_n) = 3n$.

$\overrightarrow{\gamma_m^\infty}$ has been considered on square grids, toric grids, toric king grids and toric hypergrids.

Theorem 12.

$$\left\lceil \frac{mn}{4} \right\rceil \leq \overrightarrow{\gamma_m^\infty}(P_n \Box P_m) \leq \left\lceil \frac{nm}{2} \right\rceil.$$

Theorem 13.

$$\left\lceil \frac{mn}{4} \right\rceil \leq \overrightarrow{\gamma_m^\infty}(C_n \Box C_m) \leq \left\lceil \frac{nm + 4m + 4n - 8}{3} \right\rceil.$$

When m and n are both multiples of 3, we have:

$$\overrightarrow{\gamma_m^\infty}(C_n \Box C_m) \leq \frac{nm}{3}.$$

For toric king grids, we have the following result.

Theorem 14.

$$\overrightarrow{\gamma_m^\infty}(C_n \boxtimes C_m) \leq \left\lceil \frac{nm}{5} \right\rceil + O(n+m).$$

When m and n are both multiples of 5, we have:

$$\overrightarrow{\gamma_m^\infty}(C_n \boxtimes C_m) \leq \frac{nm}{5}.$$

We also generalize Theorem 13 to toric hypergrids.

Theorem 15. $\overrightarrow{\gamma_m^\infty}(C_{n_1} \Box \ldots \Box C_{n_k}) \leq \frac{n}{k+1}$ *where n is the order of the graph and all n_i are multiples of $k+1$.*

4 Conjectures and open questions

We now enumerate some future works and open questions.

Is there a natural parameter for digraphs that is an upper bound of γ^∞ as the clique covering number is for graphs?

Give better bounds for $\overrightarrow{\gamma^\infty}$ on complete graphs, and in particular, determine if $\overrightarrow{\gamma^\infty}(K_n) = \overrightarrow{\alpha}(K_n)$ for any n.

We conjecture that the upper bounds in Theorem 15 and in special cases of Theorem 13 and 14 correspond to the exact value of $\overrightarrow{\gamma_m^\infty}$.

Study the complexity of deciding whether $\overrightarrow{\gamma_m^\infty} \leq k$ in the general case and when k is fixed.

We conjecture that $\overrightarrow{\gamma_m^\infty}(G) \leq \lceil \frac{n}{2} \rceil$ for every strongly connected digraph G.

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