FPT algorithms to recognize well covered graphs

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Abstract

Given a graph G, let $vc(G)$ and $vc^+(G)$ be the sizes of a minimum and a maximum minimal vertex covers, respectively. We say that G is well covered if $vc(G) = vc^+(G)$ (that is, all minimal vertex covers have the same size). Deciding if a graph is well covered is coNP-complete. In this paper, we obtain $O^*(2^{vc})$-time and $O^*(1.4656^{vc^+})$-time FPT algorithms to decide well coveredness, parameterized by $vc(G)$ e $vc^+(G)$, respectively, improving results of 2015 by Boria et. al. [4]. We also obtain an FPT algorithm parameterized by $\alpha(G) = n - vc(G)$ for d-degenerate graphs, which include bounded genus graphs (as planar graphs) and graphs with bounded maximum degree. Finally, we use the primeval decomposition technique to obtain a linear time algorithm for extended P_4-laden graphs and $(q, q-4)$-graphs, which is FPT parameterized by q, improving results of 2013 by Klein et al. [11].

1 Introduction

Let $G = (V, E)$ be a graph and $C, I \subseteq V$. We say that C is a vertex cover if every edge of G has an endpoint in C and that I is an independent set if every pair of distinct vertices of I are not adjacent in G. It is well known that C is a vertex cover if and only if $V - C$ is an independent set. Let $vc(G)$ be the size of a minimum vertex cover and let the independence number $\alpha(G) = n - vc(G)$ be the size of a largest independent set in G. A graph G is called *well covered* if all minimal vertex covers of G are minimum (or equivalently if all maximal independent sets are maximum). The concept of well covered graph was introduced by Plummer [12]. In well covered graphs, the greedy algorithm for producing a maximal independent set (or a minimal vertex cover) always produces a maximum independent set (and a minimum vertex cover). Unfortunately, deciding if a graph is well covered is coNP-complete even on $K_{1,4}$-free graphs [5]. In this paper, we obtain $O^*(2^{vc})$-time and $O^*(1.4656^{vc^+})$-time FPT algorithms to decide well coveredness, parameterized by $vc(G)$ e $vc^+(G)$, resp., improving results of 2015 by Boria et. al. [4]. We also obtain an FPT algorithm parameterized by $\alpha(G) = n - vc(G)$ for d-degenerate graphs, which include bounded genus graphs and graphs with bounded maximum degree. Finally, we use the primeval decomposition technique to obtain a linear time algorithm for extended P_4-laden graphs and $(q, q-4)$-graphs, which is FPT parameterized by q, improving results of 2013 by Klein et al. [11].

2 Well coveredness of graphs with few P_4's

A *cograph* is a graph with no induced P_4 [6]. A graph G is P_4-*sparse* if every set of five vertices in G induces at most one P_4 [10]. A graph G is $(q, q-4)$ for some integer $q \geq 4$ if every subset with at most q vertices induces at most $q-4$ P_4's [2]. Cographs and P_4-sparse graphs are exactly the $(4, 0)$-graphs and the $(5, 1)$-graphs. In [2], polynomial time algorithms are obtained for several optimization problems in $(q, q-4)$-graphs. A graph is extended P_4-laden if every induced subgraph with at most six vertices contains at most two induced P_4's or is $\{2K_2, C_4\}$-free. This graph class was introduced in [9]. A motivation to develop algorithms for extended P_4-laden graphs and

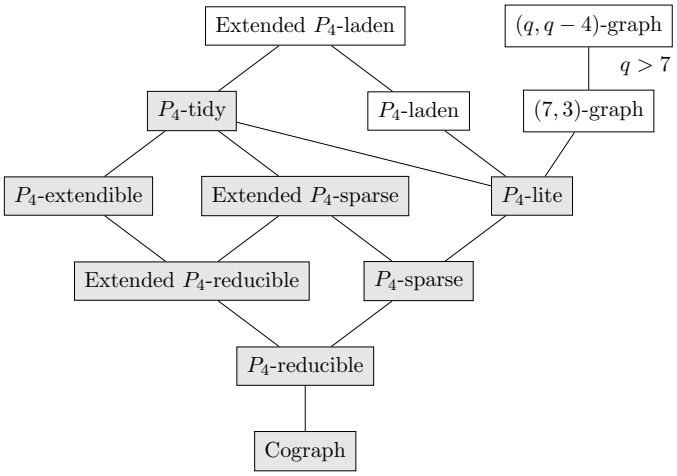


Figure 1: Hierarchy of graphs with few P_4's. In gray, the classes investigated in [11].

$(q, q-4)$-graphs lies on the fact that they are on the top of a widely studied hierarchy of classes containing many graphs with few P_4's, including cographs, P_4-sparse, P_4-lite, P_4-laden and P_4-tidy graphs. See Figure 1. In [11], Klein, Mello and Morgana obtained linear time algorithms to determine well coveredness for P_4-tidy graphs. Using the primeval decomposition of extended P_4-laden graphs and $(q, q-4)$-graphs, we extended this result to the following:

Theorem 1. *Let G be a graph and $q \geq 4$. If G is a $(q, q-4)$-graph or extended P_4-laden, we can determine well coveredness in linear time $O(2^q q^2 \cdot (m+n))$.*

3 FPT parameterized by vertex cover number $vc(G)$ and $vc^+(G)$

We say that the well coveredness problem in a graph class $\mathcal{C}$ is *fixed parameter tractable* (FPT) with some parameter $k = k(G)$ in time $O(n^c)$ (for a constant c) if deciding well coveredness is $O(f(k) \cdot n^c)$-time solvable for any graph G of $\mathcal{C}$, where f is a function depending only on $k = k(G)$. In this case, we say that the running time is $O^*(f(k))$.

In 2015, Boria et al. [4] obtained an $O^*(1.5397^{vc^+})$-time FPT algorithm to compute $vc^+(G)$ (which can be used to decide well coveredness). The following theorem improves this results.

Theorem 2. *Deciding well coveredness is $O^*(1.4656^{vc^+})$-time solvable.*

In 2015, Boria et al. [4] also obtained an $O^*(2.8284^{vc})$-time FPT algorithm to compute the maximum minimal vertex cover. In 2018, Alves et al. [1] proved explicitly that the well coveredness problem is FPT parameterized by $vc(G)$, by an FPT algorithm with time $O^*(2^{nd}) = O^*(2^{vc+2^{vc}})$, where $nd(G)$ is the neighborhood diversity of G. In the following, we improve these results.

Theorem 3. *It is possible to enumerate in time $O(2^{vc} \cdot (m+n))$ all minimal vertex covers of a graph. Consequently, there exists an $O^*(2^{vc})$-time FPT algorithm to decide well coveredness parameterized by $vc = vc(G)$.*

Sketch of the proof. Let C be a minimum vertex cover of G. Then all edges have an endpoint in C. Therefore, for every partition of C in two sets A and B ($A \cup B = C$, $A \cap B = \emptyset$), $(A \cup N(B)) \setminus B$

is a vertex cover of G if there are no edges with both endpoints in B. Moreover, for every minimal vertex cover C' of G, $A = C \cap C'$ and $B = C \setminus C'$ form a partition of C such that $C' = (A \cup N(B)) \setminus B$, since $C' \setminus C \subseteq N(B)$ (because C' is a vertex cover and is minimal). Thus, we can enumerate all minimal vertex covers of G by checking for every partition (A, B) of C if $(A \cup N(B)) \setminus B$ is a minimal vertex cover of G. Notice that verifying if a set is a minimal vertex cover can be done in time $O(m + n)$. Since there are $2^{|C|}$ partitions of C, $|C| = vc(G)$ and it is possible to obtain a minimum vertex cover C in time $O(2^{vc} \cdot (m + n))$, we are done. □

4 FPT parameterized by $\alpha(G) = n - vc(G)$

The local-treewidth [7] of a graph G is the function $ltw_G : \mathbb{N} \to \mathbb{N}$ which associates with any $r \in \mathbb{N}$ the maximum treewidth of an r-neighborhood in G. That is, $ltw_G(r) = \max_{v \in V(G)}\{tw(G[N_r(v)]\}$, where $N_r(v)$ is the set of vertices at distance at most r from v. We say that a graph class $\mathcal{C}$ has bounded local-treewidth if there is a function $f_{\mathcal{C}} : \mathbb{N} \to \mathbb{N}$ such that, for all $G \in \mathcal{C}$ and $r \in \mathbb{N}$, $ltw_G(r) \leq f_{\mathcal{C}}(r)$. It is known that graphs with bounded genus or bounded maximum degree have bounded local-treewidth [7]. In particular, a graph with maximum degree Δ has $ltw_G(r) \leq \Delta^r$ and a planar graph has $ltw_G(r) \leq 3r - 1$ [3].

Theorem 4. *The well coveredness decision problem is FPT parameterized by $\alpha(G) = n - vc(G)$ in time $O(n^2)$ for graphs with bounded local-treewidth.*

Sketch of the proof. Let $WellCov_k$ the first order formula which is true if and only if the graph G does not have two independent sets X and Y with $|X| = k$, $|Y| = k - 1$ and Y being maximal:

$$WellCov_k \ := \ \forall x_1, \ldots, x_k \ \forall y_1, \ldots, y_{k-1} \ \left(\bigwedge_{1 \leq i < j \leq k} x_i \neq x_j \right) \ \wedge \ Indep(\{x_1, \ldots, x_k\})$$

$$\rightarrow \ \neg\Big(Indep(\{y_1, \ldots, y_{k-1}\}) \ \wedge \ Maximal(\{y_1, \ldots, y_{k-1}\}\Big),$$

where $Indep(X) := \forall x, y(x \in X \wedge y \in X) \rightarrow \neg E(x, y)$ and $Maximal(X) := \forall y \exists x(y \notin X) \rightarrow (x \in X) \wedge E(x, y)$. Notice that, if G is not well covered, then there are independent sets X and Y with $2 \leq |X| \leq \alpha$, $|Y| = |X| - 1$ and Y being maximal. So, let $WellCov$ be the first order formula, which is true if and only if G is well covered:

$$WellCov \ := \ \bigwedge_{2 \leq k \leq \alpha} WellCov_k.$$

Then the well covered decision problem is first order expressible. Moreover, $WellCov$ contains at most α^2 variables and then the size of the expression $WellCov$ is a function of α. We then can apply the Frick-Grohe Theorem (see [8]) to prove that the well coveredness decision problem is FPT with parameter $\alpha(G)$ in time $O(n^2)$ for graphs with bounded local treewidth. □

We can obtain specific FPT algorithms (parameterized by $\alpha(G)$) for d-degenerate graphs, such as planar graphs, bounded genus graphs and bounded maximum degree graphs. A graph is called *d-degenerate* if every induced subgraph has a vertex with degree at most d. The *degeneracy* of a graph G is the smallest d such that G is d-degenerate. For example, outerplanar graphs, planar graphs and graphs with bounded maximum degree Δ have degeneracy at most 2, 5 and Δ, respectively.

Theorem 5. *The well coveredness decision problem is FPT parameterized by* $\alpha = \alpha(G) = n - vc(G)$ *in time* $O((d+1)^\alpha \cdot (m+n))$ *for* d*-degenerate graphs, for every* $d > 0$*. Moreover, the time is* $O(7^\alpha \cdot (m+n))$*, if* G *has bounded genus.*

Sketch of the proof. Let G be a d-degenerate graph. The algorithm uses a search tree with height $\alpha(G)$ where each node has an associated graph. The root graph is the original graph G. A leaf is a node such that its height is $\alpha(G)$ or its associated graph is empty. Let h be a non-leaf node. We branch h according to a vertex v with minimum degree in the associated graph of h. Let $N[v] = \{u_1, \ldots, u_\ell\}$, where $\ell = |N[v]| \leq d+1$. With this, the node h will have $\ell + 1$ child nodes $h_1, h_2, \ldots, h_\ell$ in the search tree. In the child node h_i $(1 \leq i \leq \ell)$, remove $N[u_i]$ from the associated graph, which is also d-degenerate. If there are two leaf nodes with different heights, return NO (since G has a maximal independent set which is not maximum and then G is not well covered). Otherwise, return YES. Notice that the tree height is at most $\alpha(G)$ and each node has at most $d+1$ child nodes. Therefore, the search tree has at most $(d+1)^\alpha$ nodes and the total time is $O((d+1)^\alpha \cdot (m+n))$, since every node takes time $O(m+n)$. □

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