Isoperimetry in integer lattices

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Abstract

The edge isoperimetric problem for a graph G is to determine, for each n, the minimum number of edges leaving any set of n vertices. In general this problem is NP-hard, but exact solutions are known in some special cases, for example when G is the usual integer lattice. We solve the edge isoperimetric problem asymptotically for every Cayley graph on $\mathbb{Z}^d$. The near-optimal shapes that we exhibit are zonotopes generated by line segments corresponding to the generators of the Cayley graph.

For every space equipped with notions of size and boundary of subsets there is a corresponding isoperimetric problem: how small can the boundary be over all subsets of a given size? For example, the classical isoperimetric theorem states that the measurable subset of $\mathbb{R}^d$ with minimum boundary for a given volume is an appropriate scaling of the unit ball. Isoperimetric problems can also be posed for graphs, where they are closely related to the phenomenon of expansion.

There are two commonly studied isoperimetric problems on graphs, corresponding to two natural definitions of the boundary of a set of vertices. Given a graph G, the *edge boundary* of a set $S \subseteq V(G)$ is

$$\partial(S) = \partial_G(S) = |\{(u,v) \in E(G) \,:\, u \in S, v \in V(G) \setminus S\}|,$$

and the *vertex boundary* is

$$\partial_v(S) = \partial_{v,G}(S) = |\{v \in V(G) \setminus S \,:\, \text{there exists } u \in S \text{ such that } (u,v) \in E(G)\}|.$$

That is, the edge boundary is the number of edges leaving S, and the vertex boundary is the number of vertices we can reach by following these edges. We write

$$\partial^*(n) = \min_{S \subseteq V(G) \,:\, |S| = n} \partial(S), \qquad \partial_v^*(n) = \min_{S \subseteq V(G) \,:\, |S| = n} \partial_v(S)$$

for the minimum size of the edge or vertex boundary over all subsets of size n. The edge (respectively, vertex) isoperimetric problem on G is to determine the function ∂^* (respectively, ∂_v^*). Solving either of these problems for a general graph G is NP-hard, but results are known in several special cases where G has a lot of structure. The approximate shape of the optimal sets for some families of graphs are listed in Table 1.

Graph	Edge-optimal shapes	Vertex-optimal shapes
$\mathcal{Q}_d$	subcubes [1, 2, 3, 4]	Hamming balls [5]
$(\mathbb{Z}^d, l_1)$	cubes (l_∞-balls) [6]	cross-polytopes (l_1-balls) [7]
$(\mathbb{Z}^d, l_\infty)$		cubes [8]

Table 1: Approximate shapes of optimal sets for isoperimetric problems on certain families of graphs.

The d-dimensional hypercube $\mathcal{Q}_d$ is the graph on vertex set $\{0,1\}^d$ with edges between those pairs of binary strings that differ in a single coordinate. With a coding theory application in mind, the edge isoperimetric problem was solved independently by mutliple authors [1, 2, 3, 4]. The corresponding vertex isoperimetric problem was solved by Harper [5].

Isoperimetric problems have also been studied for many grid-like graphs. Let $(\mathbb{Z}^d, l_1)$ be the graph on vertex set $\mathbb{Z}^d$ with edges between pairs of vertices at l_1-distance 1. Wang and Wang [7] solved the vertex isoperimetric problem on this graph and the edge isoperimetric problem was solved by Bollobás and Leader [6]. More recently, Radcliffe and Veomett [8] solved the vertex isoperimetric problem on the l_∞-grid $(\mathbb{Z}^d, l_\infty)$, where two points are adjacent if they are at l_∞-distance 1.

For each of the preceding results the authors solved the isoperimetric problem exactly. In fact, they found an ordering $v_1, v_2, \ldots$ of the vertex set such that, for each n, the set $\{v_1, \ldots, v_n\}$ has boundary of minimum size. These orderings remain consistent as the dimension d varies, in the following sense. Write G_d for either $\mathcal{Q}_d$, $(\mathbb{Z}^d, l_1)$ or $(\mathbb{Z}^d, l_\infty)$. Then viewing G_d as a subgraph of G_{d+1} in the natural way, the optimal order for G_{d+1} restricts to an optimal ordering for G_d. This allows each of the preceding results to be proved using 'compression' techniques.

Our starting point is the following question:

Question. *What is the solution to the edge isoperimetric problem on $(\mathbb{Z}^d, l_\infty)$?*

An answer to this question would fill the gap in Table 1. In fact we will prove a more general result: we will solve the edge isoperimetric problem asymptotically for every Cayley graph on $\mathbb{Z}^d$.

0.1 Cayley graphs

Let G be a group and let $\mathcal{U}$ be a generating set for G that does not contain the identity. The *Cayley graph* $G_\mathcal{U}$ has vertex set G and edge set $\{(g, ug) : g \in G, u \in \mathcal{U}\}$. We shall always take G to be $(\mathbb{Z}^d, +)$ for some d and $\mathcal{U} = \{u_1, \ldots, u_k\}$ to be a finite set of non-zero vectors. Then $G_\mathcal{U}$ is a lattice-like graph on $\mathbb{Z}^d$ in which the neighbourhood of the origin is $\mathcal{U}$ and the neighbourhood of each other vertex is obtained by translation. This construction includes both families of lattice graphs considered earlier: taking $\mathcal{U} = \{(\pm 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \pm 1)\}$ produces $(\mathbb{Z}^d, l_1)$; taking $\mathcal{U} = \{-1, 0, 1\}^d \setminus \{(0, \ldots, 0)\}$ produces $(\mathbb{Z}^d, l_\infty)$.

To state our main result we require one more piece of notation. For subsets A, B of any abelian group, the *sumset* or *Minkowski sum* of A and B is

$$A + B = \{a + b : a \in A, b \in B\}.$$

We also write

$$nA = \underbrace{A + \cdots + A}_{n \text{ times}}.$$

Theorem 1. *Let $\mathcal{U} = \{u_1, \ldots, u_k\}$ be a finite set of non-zero vectors that generate $\mathbb{Z}^d$ as a group. Let Z_0 be the sumset $\{0, u_1\} + \{0, u_2\} + \cdots + \{0, u_k\}$ and let Z be the convex hull of Z_0 in $\mathbb{R}^d$. For every $\delta > 0$, there is an $n_0 = n_0(\delta, \mathcal{U})$ such that, for every $n \geq n_0$,*

$$(1 - \delta) d \operatorname{vol}(Z)^{1/d} n^{1-1/d} \leq \partial^*_{G_\mathcal{U}}(n) \leq (1 + \delta) d \operatorname{vol}(Z)^{1/d} n^{1-1/d}.$$

The upper bound is witnessed by intersections of scaled copies of Z with $\mathbb{Z}^d$.

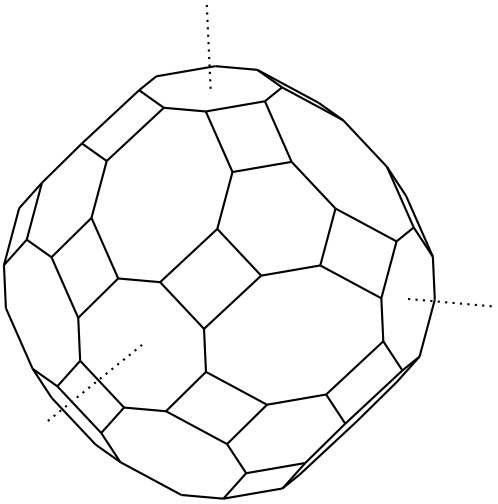


Figure 1: A near-optimal shape for the edge isoperimetric problem in $(\mathbb{Z}^3, l_\infty)$. The dotted lines represent the coordinate axes.

The set Z is a 'zonotope' in $\mathbb{R}^d$.

When $\mathcal{U} = \{(\pm 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \pm 1)\}$, we have $Z = [-1, 1]^d$, so we recover an asymptotic version of the edge isoperimetric theorem for $(\mathbb{Z}^d, l_1)$. When $\mathcal{U} = \{-1, 0, 1\}^d \setminus \{(0, \ldots, 0)\}$, corresponding to $(\mathbb{Z}^d, l_\infty)$, the zonotope Z is more complicated. For $d = 2$, it is an octagon obtained by cutting the corners off a square through points one third of the way along each side. In this case there is in fact a nested sequence of optimal sets, which interpolate between discrete versions of this octagon. This was proved by Brass [9, Theorem 3] as part of his work on the Erdős distance problem in $(\mathbb{R}^2, l_\infty)$. The zonotope Z for $d = 3$ is shown in Figure 1. When $\mathcal{U} = \{\pm(1, 0), \pm(0, 1), \pm(1, 1)\}$, $G_\mathcal{U}$ is isomorphic to the triangular lattice and Z corresponds to a regular hexagon. Harper [11, Theorem 7.2] proved that for this lattice there is an optimal ordering interpolating between regular hexagons. As far as we are aware, these are the only cases of Theorem 1 appearing in the literature. We might also ask about the vertex isoperimetric problem on these more general lattices. It turns out that this question has already been answered by Ruzsa [10].

Theorem 2 ([10])**.** *Let $\mathcal{U} = \{u_1, \ldots, u_k\}$ be a finite set of non-zero vectors that generate $\mathbb{Z}^d$ as a group. Let U be the convex hull of $\mathcal{U} \cup \{0\}$ in $\mathbb{R}^d$. For every $\delta > 0$, there is an $n_0 = n_0(\delta, \mathcal{U})$ such that, for every $n \geq n_0$,*

$$(1 - \delta) d \operatorname{vol}(U)^{1/d} n^{1-1/d} \leq \partial_{v, G_\mathcal{U}}(n) \leq (1 + \delta) d \operatorname{vol}(U)^{1/d} n^{1-1/d}.$$

The structures of the optimal sets in Theorems 1 and 2 do not seem promising for the use of compression techniques. To use compressions to prove that the shape Z in Figure 1 is optimal for the edge isoperimetric problem in $(\mathbb{Z}^3, l_\infty)$ we would like to take slices of our graph isomorphic to $(\mathbb{Z}^2, l_\infty)$ and show that the intersection of each slice with Z is itself optimal; but most cross-sections through Z are not octagons of the correct shape. To prove Theorem 2 Ruzsa instead solved a continuous approximation using tools from convex geometry, then used combinatorial methods to show that the approximation was good.

The edge isoperimetric problem has a natural continuous analogue, and the solution suggests the correct statement of Theorem 1. (Indeed, Theorem 1 has been conjectured independently by

Tsukerman and Veomett [12].) However, it is not clear that the continuous analogue is a good approximation to the original discrete problem. Instead, we will show that the edge isoperimetric problem can be related to the vertex isoperimetric problem in a related lattice.

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