Spanning trees with few branch vertices

Louis DeBiasio — **Miami University**
Allan Lo — **University of Birmingham**

Abstract

A branch vertex in a tree is a vertex of degree at least three. We prove that, for all $s \geq 1$, every connected graph on n vertices with minimum degree at least $(\frac{1}{s+3} + o(1))n$ contains a spanning tree having at most s branch vertices. Asymptotically, this is best possible and solves a problem of Flandrin, Kaiser, Kužel, Li and Ryjáček, which was originally motivated by an optimization problem in the design of optical networks.

1 Introduction

A *tree* is an acyclic connected graph and a *branch vertex* in a tree is a vertex of degree at least three. Dirac [8] proved that every graph with minimum degree at least $(n-1)/2$ contains a Hamiltonian path, i.e. a spanning tree with no branch vertices and exactly two leaves; furthermore, this is best possible as for all $n \geq 2$, there are connected graphs with minimum degree $\lceil (n-1)/2 \rceil - 1$ which have no Hamiltonian paths. This result has been generalized in many ways. In particular, Win [22] proved that if G is a connected graph on n vertices with $\delta(G) \geq (n-1)/k$, then G contains a spanning tree in which every vertex has degree at most k. Broersma and Tuinstra [1] proved that if G is a connected graph on n vertices with $\delta(G) \geq (n-k+1)/2$, then G contains a spanning tree with at most k leaves. These results are best possible for all $k \geq 2$ and when $k = 2$, they correspond to Dirac's theorem.

The problem of determining whether a connected graph contains a spanning tree with a bounded number of branch vertices, while a natural theoretical question, seems to have been first explicitly studied because of a problem related to wavelength-division multiplexing (WDM) technology in optical networks, where one wants to minimize the number of *light-splitting switches* in a *light-tree* (see [11] for a more detailed description and background). Gargano, Hell, Stacho and Vaccaro [12] showed that the problem of finding a spanning tree with the minimum number of branch vertices is NP-hard. Since then, the problem has been investigated by many authors [2, 3, 4, 6, 13, 14, 15, 18, 19, 20, 21].

A spanning tree with at most one branch vertex is called a *spider*. Gargano, Hammar, Hell, Stacho and Vaccaro [11] (also see Gargano and Hammar [10]) proved that if G is a connected graph on n vertices with $\delta(G) \geq (n-1)/3$, then G contains a spanning spider (Later Chen, Ferrara, Hu, Jacobson and Liu [5] proved the stronger result that connected graphs on $n \geq 56$ vertices with $\delta(G) \geq (n-2)/3$ contain a spanning *broom*; that is, a spanning spider obtained by joining the center of a star to an endpoint of a path). Motivated by this, Gargano et al. [11] conjectured that for all $s \geq 1$, if G is a connected graph on n vertices with $\delta(G) \geq (n-1)/(s+2)$, then G contains a spanning tree with at most s branch vertices. Later, Flandrin, Kaiser, Kužel, Li and Ryjáček [9, Problem 11] asked if the much stronger bound of $\delta(G) \geq n/(s+3) + C$ is sufficient and then Ozeki and Yamashita [16, Conjecture 30] conjectured a precise value for the constant term[1]. Note that even the approximate version of the conjecture by Flandrin et al. has not been verified for any $s \geq 1$ and the original (weaker) conjecture of Gargano et al. has not been verified for any $s \geq 2$.

[1] In both places, the conjecture is stated as a generalized Ore-type degree condition; that is, in terms of the sum of the degrees of every independent set of $s+3$ vertices, but we only state the minimum degree version here.

Conjecture 1 (Ozeki and Yamashita [16]). *For all $s \in \mathbb{N}$, if G is a connected graph on n vertices with $\delta(G) \geq \frac{n-s}{s+3}$, then G contains a spanning tree with at most s branch vertices.*

The goal of this paper is to prove Conjecture 1 asymptotically.

Theorem 1. *Let $s \in \mathbb{N}$ and let $0 < 1/n_0 \ll \gamma, 1/s$. Suppose that G is a connected graph on $n \geq n_0$ vertices with $\delta(G) \geq (\frac{1}{s+3} + \gamma)n$, then G contains a spanning tree with at most s branch vertices.*

The following example(s) show that our result is asymptotically best possible and that Conjecture 1 is best possible if true. First note that if $s+3$ divides n, then one can obtain a graph G on n vertices with $\delta(G) = \frac{n}{s+3} - 1$ which contains no spanning tree with at most s branch vertices by identifying each vertex of a path on $s+3$ vertices with a complete graph on $\frac{n}{s+3}$ vertices since the $s+1$ internal vertices of the path will be branch vertices in every spanning tree.

Example 1. *For all $s, m \in \mathbb{Z}^+$, there exists a connected graph G on $n = (s+3)m - 2$ vertices with $\delta(G) = \frac{n-s-1}{s+3}$ such that every spanning tree of G has more than s branch vertices.*

Proof. Let H_1 be a graph on $2m-1$ vertices obtained from two copies of K_m by identifying a vertex v and let H_2 be the graph obtained by taking the join of an independent set on m vertices with a K_{m-1}. Let $n = (s+3)m - 2$ and let G be a graph obtained from a path $P = b_1 b_2 \dots b_{s+1}$ on $s+1$ vertices by identifying for each $2 \leq i \leq s$, the vertex b_i with a vertex of a K_m and for $j \in \{1, s+1\}$, either by identifying b_j with the cut vertex of H_1, or by identifying b_j with some vertex in the smaller side of H_2; that is, adding some edge from b_2 (if $j = 1$) or b_s (if $j = s+1$) to the smaller side of H_2. It is clear that $\delta(G) = m - 1 = \frac{n-s-1}{s+3}$ and for any spanning tree T, each b_i will be a branch vertex in T.

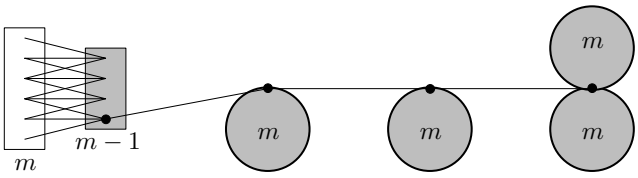


Figure 1: An example for the case $s = 3$.

□

2 Overview of the proof

Our proof splits into two main parts. First we show that if G is a graph with minimum degree at least $(1/r + \gamma)n$, then we can find a partition of $V(G)$ into at most $r-1$ parts $\{V_1, \dots, V_k\}$ having the property that for each i, $G[V_i]$ has no sparse cuts and most vertices in V_i have degree at least $(1/r + \gamma/2)n$ in $G[V_i]$ while all other vertices in V_i have linear minimum degree in $G[V_i]$. Let us say that we have partitioned G into "robust" subgraphs.

The second part of the proof focuses on these so-called robust subgraphs obtained above. Let $t \geq 1$ and let G be a graph on n vertices with linear minimum degree having no sparse cuts in which most of the vertices have degree at least $(\frac{1}{t+3} + \gamma)n$. We will show that not only does G contains a spanning tree with at most t branch vertices, but G contains a cycle C and a set $K \subseteq V(C)$ with $|K| \leq t$ such that for all $v \in V(G) \setminus V(C)$, v has a neighbor in K. It is clear that such a structure, which we call a "star-cycle", contains a spanning tree with at most t branch vertices.

The real heart of the proof lies in finding these spanning star-cycles in the robust subgraphs. It is now standard in spanning subgraph problems to use Szémerédi's regularity lemma to reduce the problem to finding a simpler structure in the so-called reduced graph. For instance, if one were looking for a Hamiltonian cycle, it would be natural to apply regularity and prove that the reduced graph is connected and contains a perfect matching. In our case, the simpler structure that we wish to find is a collection of vertex-disjoint edges and stars which we call a "star-matching." Unfortunately it may not be sufficient to find a star-matching in the reduced graph, as this may not correspond to the desired star-cycle in the original graph (it is possible that every star-cycle in the original graph has unbounded maximum degree, a situation for which the regularity lemma is unequipped to deal with). To get around this issue, we introduce a more complex structure in place of the reduced graph, called the "fractional-random-reduced-graph". The fractional-random-reduced-graph will retain more of the information about the original graph and therefore make it possible to turn a spanning star-matching in the fractional-random-reduced-graph into a nearly spanning star-cycle in the original graph. To get from a nearly spanning star-cycle to a spanning star-cycle, we use the now standard absorbing method of Rödl, Ruciński and Szemerédi [17] in a form proved by the first author and Nelsen [7].

Finally, to combine the two parts of the proof, we start with a connected graph G having minimum degree at least $(\frac{1}{s+3} + \gamma)n$. We obtain a robust partition of G and inside each part of the partition we find a star-cycle having the correct number of stars depending on the relative degrees inside that part. Then we use the connectivity of G to find edges connecting the spanning star-cycles from each part of the partition. The minimum degree of G will put bounds on the number of parts of the partition and the relative degrees inside those parts in such a way that the obtained spanning tree has at most s branch vertices.

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