Every planar graph of maximum degree 8 is 10-choosable

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Abstract

Total coloring is a variant of vertex/edge coloring where both vertices and edges are to be colored. In this work, we investigate the list version of this problem. A graph is totally k-choosable if for any list assignment of k colors to each vertex and each edge, we can extract a proper total coloring. In this setting, a graph of maximum degree Δ needs at least $\Delta + 1$ colors. For a planar graph, Borodin proved in 1989 that $\Delta+2$ colors suffice when Δ is at least 9. Similar results hold for edge-choosability with one color less. Recently, the first author proved that for edge-choosability, $\Delta + 1$ colors are sufficient when Δ is only 8. We explain here how to extend both results to the setting of total choosability for planar graphs of maximum degree 8.

1 Introduction

Coloring a graph G consists in assigning colors to elements of G (vertices, edges, ...) in such a way that the coloring is proper, i.e., two adjacent or incident elements receive different colors. Historically, the case of vertex coloring was first to be considered. For this kind of coloring, the minimal number of colors needed to color a graph G is denoted by $\chi(G)$. While computing this number is NP-complete in general, several bounds exist for sparse graphs, the most famous one being the four-color theorem, stating that $\chi(G) \leqslant 4$ for every planar graph G.

In [7], Vizing introduced a generalization of coloring problems called choosability. The idea is to assign lists of colors to every element of a graph G, and to look for a proper coloring of G where each element receives a color from its list. The parameter we minimize is no longer the total number of colors (since it may depend on the size of the graph). We rather consider the minimal size of lists required to ensure that a proper coloring can be extracted from every assignment of large enough lists. For vertex-choosability, this parameter is denoted by χ_ℓ.

By considering the constant assignment (i.e., all elements receive the same list), we get that $\chi_\ell(G) \geqslant \chi(G)$ for every graph G. Moreover, it is known that this inequality may be strict, even for planar graphs (for example, the complete bipartite graph $K_{2,4}$ satisfies $\chi(K_{2,4}) = 2$ and $\chi_\ell(K_{2,4}) = 3$).

In this work, we focus on edge coloring and total coloring (when we color vertices and edges). For these problems, the situation seems to be quite different since it is conjectured in [6] that the corresponding parameters, χ' and χ'', are the same as their counterparts, χ'_ℓ and χ''_ℓ, for choosability.

Conjecture 1. *For every graph G, we have $\chi'(G) = \chi'_\ell(G)$ and $\chi''(G) = \chi''_\ell(G)$.*

By looking at the neighborhood of a vertex u of maximum degree in G, we obtain a simple lower bound on $\chi'(G)$ and $\chi''(G)$. Indeed, all the edges incident with u have to receive different colors, and thus $\chi'(G) \geqslant \Delta(G)$ and $\chi''(G) \geqslant \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G.

A celebrated theorem of Vizing states that $\chi'(G) \leqslant \Delta(G)+1$ for every graph G. While deciding whether $\chi'(G)$ is $\Delta(G)$ or $\Delta(G)+1$ is an NP-complete problem, the difference between the upper and lower bounds is only 1. Another conjecture states that the situation is the same for the total coloring and for the list variants:

Conjecture 2. *For every graph G, we have $\chi'_\ell(G) \leqslant \Delta(G)+1$ and $\chi''(G) \leqslant \chi''_\ell(G) \leqslant \Delta(G)+2$.*

These two conjectures were solved for graphs with small maximum degree by directly constructing the coloring. However, they remain open for higher maximum degrees. For sparse graphs (like planar ones), we have a useful tool: the discharging method. This method was first introduced in 1904 by Wernicke [8]. Since then, many results of graph theory have been proven using this method, the main one being the four-color theorem [1].

With the discharging approach, both conjectures 1 and 2 have been confirmed for planar graphs of high enough maximum degree. Just to cite a few results, for every planar graph G of maximum degree Δ, we have $\chi'_\ell(G) \leqslant \Delta+1$ if $\Delta \geqslant 9$ [5], and $\chi'_\ell(G) = \chi'(G) = \Delta$ if $\Delta \geqslant 12$ [6]. For total coloring, similar bounds hold: $\chi''_\ell(G) \leqslant \Delta+2$ if $\Delta \geqslant 9$ [4] and $\chi''_\ell(G) = \chi''(G) = \Delta+1$ if $\Delta \geqslant 12$ [6].

Recently, using the discharging method, the first author proved that the bound $\chi'_\ell \leqslant \Delta+1$ holds for planar graphs of maximum degree 8 [3]. Following the similarity between the bounds for edge and total coloring, we investigate the total coloring problem on planar graphs of maximum degree 8 and prove the following result.

Theorem 1. *Every planar graph of maximum degree $\Delta \geqslant 8$ is $(\Delta+2)$-choosable.*

In the following, we explain how to use the discharging method to prove this theorem. We also present more specifically some of the tools we use.

2 Main tools

As explained previously, our approach uses the discharging method. Like most of proofs using discharging, ours is based on two main results of different kind. The first one is where we use discharging arguments: we show that every planar graph of maximum degree 8 contains some specific configurations. We say that the set of these configurations is *unavoidable*. Then, in a second part, we prove that each of these configurations is *reducible*, meaning that if a graph G contains a configuration C and $G \setminus C$ is totally 10-choosable, then so is G.

By proving these two assertions, we actually design a custom induction scheme. Indeed, unavoidability of a set S ensures that every planar graph of maximum degree 8 can be constructed from the empty graph by successively adding some configurations of S. This proves that the underlying induction scheme is complete.

Similarly, proving that every configuration of S is reducible is exactly the same as proving the induction step for total 10-choosability. Combining these two properties finally proves by induction our result. We now give the main ideas we used to prove unavoidability and reducibility.

2.1 Proving unavoidability

This part is where we make an extensive use of the discharging method. We proceed as follows: we assume there exists a planar graph of maximum degree 8 containing no configuration of a given set S, and derive a contradiction.

To this end, we assign to each element of G (vertices, edges and faces) a weight ω. Then, we design some rules to transfer weights while preserving the total weight. We end up with a new distribution of weights denoted by ω'. Note that since the total weight is preserved by applying the rules, we must have $\sum \omega = \sum \omega'$ where both sums are indexed by all the elements of G. We finally use the fact that G contains none of the configurations of S to prove that $\sum \omega \neq \sum \omega'$, which directly leads to a contradiction.

2.2 Reducing each configuration

To prove that a configuration C is reducible, we may consider the following generic approach. We take a graph G containing C and such that $G \setminus C$ is totally 10-choosable. For any list assignment L of G, we can find an L-coloring of $G \setminus C$. We then construct a new assignment L' of C: for every element x of G, $L'(x)$ is obtained by removing from $L(x)$ the colors of every element adjacent to or incident with x. We then show that C is L'-colorable, and this ensures that G is L-colorable.

This approach is very simple to apply. However, it often requires to distinguish many cases according to the behavior of the list assignment L'. The obtained proofs may then quickly become fastidious to read. To avoid this issue, we use two other approaches. The first one uses a powerful result of combinatorics from [2] that we state below.

Theorem 2 (Combinatorial Nullstellensatz)**.** *Let $\mathbb{K}$ be a field, and $P \in \mathbb{K}[X_1, \ldots, X_n]$ a multivariate polynomial. Let $X_1^{a_1} \cdots X_n^{a_n}$ be a monomial with a non-zero coefficient in P, and of maximal degree. Then, for any family $S_1, \ldots, S_n$ of subsets of $\mathbb{K}$ satisfying $|S_i| > a_i$ for $i = 1, \ldots, n$, there exists a non-vanishing value of P in $S_1 \cdots S_n$.*

At first glance, this result has nothing to do with graph coloring. However, as shown in [2], it has a lot of consequences in graph theory. For total choosability, as we will recall, we may use this theorem to reduce the problem of L'-coloring C to finding a monomial with good properties with respect to a suitable polynomial.

Denote by $x_1, \ldots, x_n$ the elements of C we need to L'-color. We consider the polynomial ring $\mathbb{Q}[X_1, \ldots, X_n]$, with a variable X_i for each element x_i of C ($i = 1, \ldots, n$). We then consider the polynomial P defined as the product of all $(X_i - X_j)$ where $i < j$ and x_i, x_j are adjacent or incident elements of G. Observe that if we evaluate X_i to a value c_i in $L'(x_i)$ for every $i \in [1, n]$, then $P(c_1, \ldots, c_n) = 0$ if and only if the L'-coloring $c_1, \ldots, c_n$ is not proper. Therefore, we are looking for a non-vanishing value $(c_1, \ldots, c_n)$ for P, where each c_i lies in $L'(x_i)$, $i \in [1, n]$.

Using Theorem 2, finding an L'-coloring of C boils down to finding a monomial $X_1^{a_1} \cdots X_n^{a_n}$ of degree $\deg(P)$ such that $a_i < |L'(x_i)|$ for each $i \in [1, n]$, and a non-zero coefficient in P.

This approach leads to much easier to verify proofs. However, its downside is that we lose every graph-related intuition on how to find an L'-coloring of C. Moreover, due to the size of the polynomials, finding a suitable monomial by hand is unlikely, as well as checking if the whole proof works. Thus, we often rely on a computer, when using this approach.

We also use a third approach that was introduced in [3]. The idea is similar to our first naive approach, excepted that we allow to change the color of some elements of $G \setminus C$. First, we prove that we can find an L'-coloring unless some conditions on L' holds. Then, we show that we can change the color of some vertices in $G \setminus C$ to ensure that these conditions are not satisfied. This gives much shorter proofs along with more insight on how to find an L-coloring of G.

3 Conclusion and future work

We present a discharging proof that planar graphs of maximum degree 8 are totally 10-choosable. This transposes the result from [3] to the total coloring problem, and extends a result from [4]. While the approach we use is similar to the one used in [3], the proof does not directly translate from edge coloring to total coloring. For example, in [3], two adjacent vertices of degree 5 form a reducible configuration, while this does not hold anymore for total choosability. Thus we needed to introduce more configurations: our proof uses 19 configurations whose sizes are similar to the 9 presented in [3]. Moreover, while the recoloring approach was already introduced in [3], we also extended the arguments to develop a more generic framework for recoloration.

As it is often the case with the discharging method, our proof gives a linear algorithm to find an L-coloring, for every assignment of lists of size at least 10. Indeed, using discharging to prove unavoidability gives directly a linear algorithm to find every reducible configurations of a given graph. Then, all the reduction proofs are constructive and give a constant time algorithm to color each configuration, unless we use the Combinatorial Nullstellensatz. However, in this case, since the sizes of lists and of configurations are bounded, we may find in constant time a coloring of the considered configuration by enumerating every possible list assignments.

We believe that the discharging method may extend our result to the case of planar graphs of maximum degree 7. However, we may need to consider much bigger configurations. Therefore, extending our result to this case may need more advanced discharging, and much more work to find and reduce configurations.

Our approaches for reducing configurations also makes arise several questions. For example, the recoloring approach comes from [3] where it is used to recolor sets of pairwise incident elements. When extending this idea, we did not get rid of this restriction, and we do not know if this can be done. Most of the questions arise for the Combinatorial Nullstellensatz approach. For example, we do not know whether this approach gives an equivalence. Indeed, we know that if we can find a suitable monomial, then the configuration is reducible. However, we do not know if the converse statement holds, nor if we can deduce a suitable monomial from some reduction proof. Moreover, this approach is not constructive a priori, and we do not know if we can deduce a coloring from the data of a suitable monomial.

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