Graph coloring reduction methods and application to McDiarmid-Reed's Conjecture

Olivier Togni — **Le2I, Université Bourgogne Franche-Comté, France**
Jean-Claude Godin — **Université du Sud Toulon-Var, France**

Abstract

We define general reduction tools for (a, b)-coloring of graphs where $2 \leq a/b \leq 3$. The utility of these tools is exemplified on finite triangle-free induced subgraphs of the triangular lattice. Computations on millions of such graphs generated randomly show that our tools allow to find (in linear time) a $(9, 4)$-coloring for each of them.

A (a, b)-*coloring* of a graph G is a mapping which associates to each vertex a set of b colors from a set of a colors in such a way that adjacent vertices get disjoints sets of colors. Such type of coloring is in relation with weighted colorings (in which the number of colors to assign to each vertex is variable) which have application for frequency allocation in cellular networks [3].

In this paper, we define reducible configurations for (a, b)-coloring when $2 \leq a/b \leq 3$, that may allow to prove that some graphs are (a, b)-colorable. In particular, we apply our reduction tools on finite induced subgraphs of the triangular lattice, called *hexagonal graphs* in this paper, in order to try to solve the following conjecture due to McDiarmid and Reed:

Conjecture 1 (McDiarmid-Reed [1])**.** *Every hexagonal graph is* $(9, 4)$*-colorable.*

1 Handle reductions

A *handle* $P(n)$ of length n in a graph G is a path of length n that is an induced subgraph of G with vertices of degree two in $P(n)$ having the same degree in G. The *interior* $\mathsf{int}(H)$ of a handle H is the set of vertices of degree 2 of H. A *parity handle (or P-handle)* $PP(n)$ in a graph G is a handle $P(n)$ with the additional property that there exists another path of length $m \leq \ell$, of the same parity than ℓ in $G - \mathsf{int}(P(n))$, between the two end-vertices of $P(n)$. An *S-handle* $S(n_1, n_2, n_3)$ in a graph G is a handle of length n_1 such that one of its end-vertices has degree 3 in G and is also the end-vertex of two other handles of length n_2 and n_3. An *H-handle* $H(n_1, n_2, n, n_3, n_4)$ in a graph G is a handle of length n such that its two endvertices are of degree three in G and one of them is also the end-vertex of two other handles of lenght n_1 and n_2 and the other end vertex is also the end-vertex of two handles of length n_3 and n_4. Examples of handles are illustrated in Figure 1. Due to symetry reasons, we will consider only S-handles with $n_1 \geq n_2 \geq n_3$ and H-handles with $n_1 \leq n_2 \leq n$ and $n_4 \leq n_3 \leq n$. Note that extremities of a handle may be the same vertices (hence a handle may induce a cycle in the graph).

A handle H is (a, b)-*reducible* in a graph G if any (a, b)-coloring of $G - \mathsf{int}(H)$ can be extended to a (a, b)-coloring of G, i.e., whatever the colors of the end-vertices of the handle, there always exists a (a, b)-coloring of $\mathsf{int}(H)$.

For a graph G and a family $\mathcal{F}$ of handles in G, we define $\mathsf{core}_{\mathcal{F}}(G)$ as any (induced) subgraph obtained after successively removing vertices of degree 0 and 1 and vertices of $\mathsf{int}(H)$ for each handle $H \in \mathcal{F}$ until no more degree 0 or 1 vertex nor handle of $\mathcal{F}$ remains. It can be shown that the core is in fact unique, i.e., whatever the order of the reductions made, it will end with the same graph.

By the definition of reducibility, we immediatly have the following result:

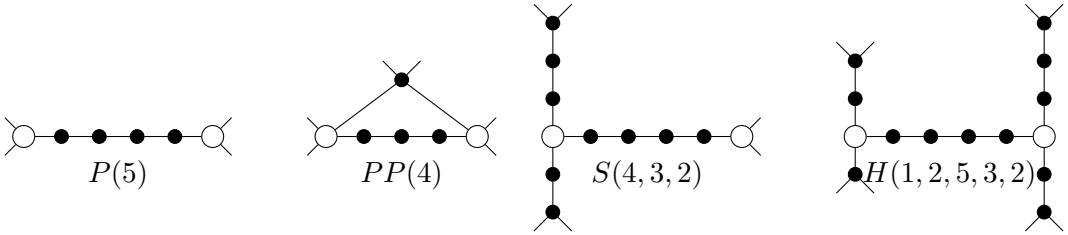


Figure 1: Examples of handles in a graph, from left to right: handle, parity handle , S-handle and H-handle.

Theorem 2. *For any graph G and any familiy $\mathcal{F}$ of (a,b)-reducible handles in G,*

$$G \text{ } (a,b)\text{-colorable} \Leftrightarrow \mathsf{core}_{\mathcal{F}}(G) \text{ } (a,b)\text{-colorable.}$$

For any real x, let $\mathsf{Even}(x)$ be the minimum even integer m such that $m \geq x$. When speaking about smallest reducible handles, it refers to the natural ordering on the vectors of integers. The following reducibility results are given without proof due to space constraints.

Theorem 3. *For any graph G and any integers b, e such that $b \geq 2$ and $e < b$, any handle $P(n)$ with $n \geq \mathsf{Even}(2b/e)$ is $(2b+e,b)$-reducible in G and any parity handle $PP(n)$ with $n \geq 3$ is $(2b+e,b)$-reducible in G.*

Theorem 4. *For any graph G and any integers b, e, k such that $b \geq 2, e < b$, and $k \leq b$, $S(\mathsf{Even}(2b/e)-1,2,1)$ is the smallest $(2b+e,b)$-reducible S-handle and $S(2b-k,k,k)$ is the smallest $(2b+1,b)$-reducible S-handle in G.*

Theorem 5. *For any graph G and any integers b, e such that $b \geq 2$ and $e < b$, $H(1,2,\mathsf{Even}(2b/e)-2,2,1)$ is the smallest $(2b+e,b)$-reducible H-handle and the following H-handles are the smallest $(2b+1,b)$-reducible H-handles in G:*

- $H(2,2,2b-3,2,2)$, $H(1,2,2b-3,3,2)$, $H(1,4,2b-3,2,2)$;
- $H(2,3,2b-4,3,2)$, $H(2,2,2b-4,4,2)$, $H(2,2,2b-4,3,3)$, $H(1,2,2b-4,4,3)$, $H(1,4,2b-4,3,3)$.

2 Multicoloring hexagonal graphs

In a similar way than the method used by Havet [2], i.e., starting from a degree 3 vertex in a 'corner' of the graph and exploring the configurations around it for proving that a handle from $\mathcal{F}$ is present, we can prove the following:

Theorem 6. *Let H be a hexagonal graph. For each of the following three families of handles we have $\mathsf{core}_{\mathcal{F}}(H) = \emptyset$:*

1. $\mathcal{F} = \{P(2)\}$;
2. $\mathcal{F} = \{P(4), PP(3)\}$;

3. $\mathcal{F} = \{P(6), PP(3), PP(4), PP(5), S(5,2,1), H(1,2,4,2,1)\}$.

This theorem along with Theorems 3, 4, 5 allow to prove the known results [2] for (a,b)-colorability of hexagonal graphs for $b = 1, 2$ and 3 in a unified way:

Corollary 7. *Any hexagonal graph is* $(3,1)$*-colorable,* $(5,2)$*-colorable and* $(7,3)$*-colorable.*

For $a = 9$ and $b = 4$, it allows to prove simply the result of Witkowski and Žerovnik [4], where a corner is a vertex of degree 2 or 3 adjacent to at least two vertices of non opposite directions:

Corollary 8. *Any hexagonal graph without adjacent corners is* $(9,4)$*-colorable.*

For any hexagonal graph H, consider the following set of handles:
$\mathcal{F}_{9,4} = \{P(8), PP(3), PP(4), PP(5), PP(6), PP(7)\} \cup \{S(7,2,1), S(6,2,2), S(5,3,3), S(4,4,4)\} \cup$
$\{H(1,2,6,2,1), H(1,2,5,3,2), H(1,4,5,2,2), H(2,2,5,2,2), H(2,2,4,4,2), H(2,2,4,3,3),$
$H(1,2,4,4,3), H(2,3,4,3,2), H(1,4,4,3,3)\}$.

Note that, by Theorems 3, 4, 5, the handles of $\mathcal{F}_{9,4}$ are all $(9,4)$-reducible. Moreover, all the handles of $\mathcal{F}_{9,4}$ are necessary since we have examples of hexagonal graphs for which the core is not empty if we remove one of the handle from $\mathcal{F}_{9,4}$. Figure 2 shows three examples of hexagonal graphs that only possess H-handles (the graphs on the left and right contains only handles $H(2,2,4,4,2)$ and the one on the center only handles $H(1,2,6,2,1)$.

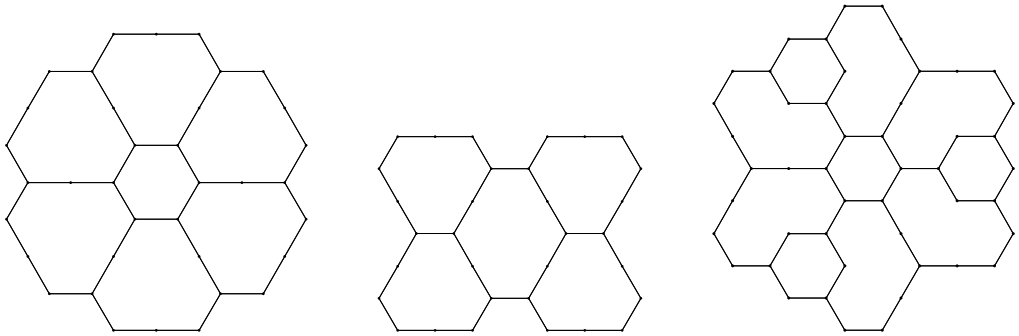


Figure 2: Three hexagonal graphs that only have $(9,4)$-reducible H-handles.

We have performed computational experiments for testing the reduction tools defined in the previous section on hexagonal graphs for finding a $(9,4)$-coloring. The algorithms were coded in C++ and ran on a Intel Xeon CPU at 2.67 GHz. The graphs are generated randomly on a grid of size $\ell \times h$ by choosing randomly the coordinates of a point and testing if the corresponding vertex can be added to the graph without creating a triangle (repeated $10\ell h$ times). In order to obtain the 'harder' instances of hexagonal graphs, we then do a final pass in which we consider the points of the grid in order and test if they can be added to the graph. Hence the graph obtained are maximal triangle-free subgraphs of the triangular lattice, i.e., no point in the area can be added without creating a triangle. The reduction algorithm consists in testing, for each vertex x in sequence, if x is the end-vertex of a handle from $\mathcal{F}_{9,4}$. The results are reported on Table 1 in which the first column indicates the grid size, the second column reports the number of graphs generated and tested, the third column indicates the mean density, i.e., the mean ratio of degree-3 vertices, the fourth one indicates mean order of the graphs tested, the fifth one reports the ratio of graphs that require at least a reduction $H(n_1, n_2, n, n_3, n_4)$. Column 6 gives the time needed to test all the graphs and column 7 reports the number of graphs for which the core is not empty.

Grid size $\ell \times \ell$	number of graphs tested	mean density	mean order	H-ratio (per 1000)	computation time (s)	number of graphs with $\mathsf{core}_{\mathcal{F}_{9,4}} \neq \emptyset$
10	10000000	20	35	0	732	0
12	10000000	23	56	0.0006	1020	0
16	10000000	27	108	0.0016	2000	0
20	10000000	29	176	0.0062	3030	0
30	10000000	32	416	0.0335	7296	0
40	10000000	33	755	0.0963	13740	0
50	1000000	34	1193	0.184	22140	0
75	1000000	35	2721	0.528	5508	0
100	1000000	36	4868	1.118	9792	0
150	1000000	36	11016	2.815	22960	0
200	100000	36	19639	5.12	4278	0
500	20000	37	123345	36.65	6036	0
1000	1000	37	494129	142	1329	0
1500	1000	37	1112383	283	3243	0

Table 1: Summary of the computations.

Our computations show that on thousand millions of graphs on grid sizes ranging from 10×10 to 1500×1500, every graph has been reduced completely, i.e., its core is empty. Moreover, the ratio of hexagonal graphs that cannot be reduced completely without using H-handles is quite small (although this ratio grow with the order of the graph, see column 5 of Table 1).

With these new reductions, we can state the following equivalent form of Conjecture 1:

Conjecture 9. *For every hexagonal graph G, $\mathsf{core}_{\mathcal{F}_{9,4}}(G)$ is $(9,4)$-colorable.*

However, the $(9,4)$-reducible handles considered so far are not sufficient to prove McDiarmid and Reed's conjecture since our program, ran on hexagonal graphs generated from the flower of Figure 2 (by adding vertices randomly), found few examples for which $\mathsf{core}_{\mathcal{F}_{9,4}}$ is not empty. For each we have to define a new type of handle with constraints on the distances between some end-vertices (that in turn, induce constraints on the colors of theses vertices). One of such configurations is the *hexagonal handle* $H^*(2,2,n,2,2)$ which is a handle $P(n)$ with a length 6 cycle attached to each of its two end-vertices (hence it is a H-handle $H(2,2,n,2,2)$ with a path of length two between the two end-vertices of the paths of length two attached to each extremity of the central path).

References

[1] C. McDiarmid and B. Reed. Channel assignment and weighted coloring. *Networks*, 36, (2000), 114-117.

[2] F. Havet, Channel assignment and multicolouring of the induced subgraphs of the triangular lattice. *Discrete Math.* 233, (2001), 219-233.

[3] J. Janssen, D. Krizanc, L. Narayanan, S. Shende, Distributed Online Frequency Assignment in Cellular Networks. *Journal of Algorithms*, Proceedings, 36(2) (2000), 119-151.

[4] R. Witkowski, J. Žerovnik, Proof of McDiarmid-Reed conjecture for a subclass of hexagonal graphs. *Utilitas Mathematica*, 105 (2017), 191–206.