Online graph coloring with bichromatic exchanges

Marc Heinrich — **LIRIS, Université Lyon 1**
Sylvain Gravier — **CNRS - Institut Fourier, Maths à Modeler, Grenoble**

Abstract

Greedy algorithms for the graph coloring problem require a large number of colors, even for very simple classes of graphs. For example, Gyárfás and Lehel proved that any greedy algorithm coloring trees requires $\Omega(\log n)$ colors in the worst case. We consider a variation of the First-fit algorithm in which the algorithm is allowed to make modifications to previously colored vertices by performing a polynomial number of local bichromatic exchanges. We show that such algorithms can be used to find in polynomial time an optimal coloring in the case of bipartite graphs, chordal graphs and outerplanar graphs. We also show that it can find in polynomial time a coloring of general planar graphs with $O(\log \Delta)$ colors, where Δ is the maximum degree of the graph. The question of whether planar graphs can be colored by a polynomial time online algorithm with bichromatic exchanges using only a constant number of colors is still open.

Keywords: online algorithms, graph coloring, bichromatic exchange, kempe chain.

1 Introduction

Online algorithms are a class of algorithms reading their input sequentially. In the case of graph problems, this usually means that the vertices of the graph arrive one by one. As the formal definition of greedy, online, and sequential algorithms is not completely fixed, we start by by precising the convention we use here. In an online algorithm, for each new vertex, the algorithm must adapt a partial solution of the problem on the graph without the new vertex into a solution for the whole graph. For the graph coloring coloring problem, this means that, if G is the graph and v the newly added vertex, the algorithm must transform a coloring of $G - v$ into a coloring of G.

A special class of online coloring algorithms which received a lot of attention is the class of greedy coloring algorithms, where the algorithm is not allowed to change his previous choices. The algorithm must assign a color to each new vertex that is different from the colors of its neighbors. Vertices colored at an earlier step cannot be recolored. The most common greedy algorithm for graph coloring is First-fit , where the color of the new vertex is taken to be the smallest color not already present in its neighborhood. These greedy algorithms are usually studied in a setting where the order on the vertices of the graph is arbitrary. The performance of these algorithms, is measured by the number of colors used for the worst case ordering.

A similar type of algorithms are sequential algorithms. These algorithms first decide on an ordering of the vertices, and then, apply an online algorithm to color the graph according to this ordering. In this case the ordering is chosen by the algorithm.

The greedy graph coloring problem is a widely studied subject. For general graphs, a randomized greedy algorithm finding an $O(\frac{n}{\log n})$ approximation was devised in [11] while there is a lower bound of $\Omega(\frac{n}{\log^2 n})$ on the approximation ratio of any greedy algorithm. An important effort has been directed at studying the performance on usual graph classes. A greedy algorithm with an approximation ratio of $O(\log n)$ was shown for trees [7], bipartite graphs [16], planar and chordal graphs [12]. On the other hand, this approximation ratio was shown to be optimal for these graph classes [2, 1, 16]. The lower bounds even holds for randomized algorithms, algorithms using a small reodering buffer, or algorithm allowed to look at a few future inputs before making a choice [1].

Online algorithms with constant approximation ratio exists for other classes of graphs e.g. interval graphs [13, 15, 20] and disk intersection graphs [4, 5, 1].

Since the approximation ratio of greedy algorithms is quite large even for some simple classes of graphs, it is natural to look at more general algorithms. In this article, we consider online algorithms which are allowed to change the color of previous vertices only by making local bichromatic exchanges. A bichromatic exchange (also called Kempe change) consists in swapping the colors of the vertices of a maximal connected 2-colored subgraph of G. Informally this recoloring is done by selecting a vertex u say colored a, and changing its color to a new color b. To keep the coloring proper, all its neighbors colored b are recolored a, and so on. When the selected vertex u is a neighbor of some vertex v, we will say that the bichromatic exchange is local to v. An example of bichromatic exchange is given in Figure 1.

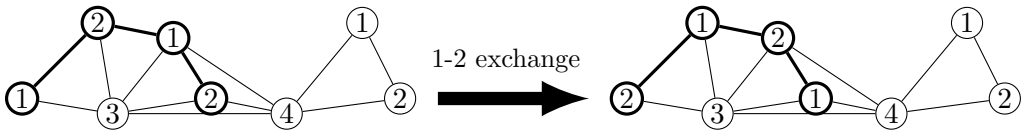


Figure 1: Example of bichromatic exchange.

We consider online algorithms with bichromatic exchanges. For each new vertex v, the algorithm can perform bichomatic exchanges which are local to the new vertex v. By applying these transformations, the algorithm might be able to remove one color from the neighborhood of v, and use this color for v.

The choice of bichromatic exchanges as a way to recolor the graph is quite natural. It was first considered in Kempe's failed attempt at proving the four color theorem, and was used later to prove Vizing's theorem [22]. More recently, sequential algorithm using bichromatic exchange were considered in [17] to color a special subclass of perfect graphs, and in [8, 9, 10, 21, 23] using more complex recolorings. In this context, bichromatic exchanges are used to locally modify an existing coloring in order to color a new vertex.

Bichromatic exchanges were also considered in the context of graph recoloring problem for planar graphs [18, 19], K_5-minor free graphs [14] and d-regular graphs [6, 3]. Given two input colorings, the recoloring problem is to decide whether you can transform the first one into the second one by applying a sequence of basic recoloring operations.

2 Results

We study online algorithms with bichromatic exchanges on several classes of graphs. For simple classes of graphs, these algorithms allow to compute optimal colorings.

Theorem 1. *There are polynomial time online coloring algorithms with bichromatic exchanges using an optimal number of colors for the following classes of graphs: bipartite graphs, chordal graphs, and outer-planar graphs. These algorithm perform at most Δ bichromatic exchanges for each new vertex in the graph.*

The main idea in all three cases is to find a nice ordering of the vertices in the neighborhood of v. Then, these vertices are recolored one by one according to this ordering. The ordering ensures that at any step, previously recolored vertices do not change color. Any ordering works in the

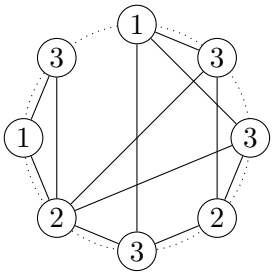


Figure 2: Example of circular drawing of a graph with an intersection compatible coloring.

bipartite case. We use an ordering from the outer-planar embedding in the outer-planar case, and a simplicial ordering for chordal graphs. A more complicated case is planar graphs, for which we can show the following:

Theorem 2. *There is a polynomial time online algorithm with bichromatic exchanges coloring any planar graph with $O(\log(\Delta))$ colors. The algorithm performs at most $\frac{\Delta^2}{2}$ bichromatic exchanges for each new vertex.*

The question whether we can achieve a constant number of colors is still open. This result is proved by looking at an auxiliary problem. Given a graph G, with a (not necessarily planar) drawing of G, a proper coloring c of G is *intersection compatible* if for any two crossing edges (u_1, v_1) and (u_2, v_2), the two sets $\{c(u_1), c(v_1)\}$ and $\{c(u_2), c(v_2)\}$ intersect. A *circular drawing* of G is a drawing of G such that all the vertices are represented by points on a circle, and the paths representing the edges are straight line segments. An example of a circular drawing of a graph with an intersection compatible coloring is given in Figure 2.

These new notions are related to the original problem by the following construction. From the planar graph G, with a coloring of $G-v$, we can construct a graph G' by keeping only the neighbors of v, and adding an edge between two vertices u_1 and u_2 with different colors if there is a bicolored path between these two vertices. The drawing of G gives a circular drawing of G', and since G is planar, two edges that cross must share a common color. Indeed, these two edges correspond to two crossing bicolored paths, and two crossing bicolored paths must share at least one color.

The algorithm for Theorem 2 consists in repeatedly applying a greedy procedure. When this procedure ends, we have the property that the coloring of G' is a grundy coloring: every vertex colored i has at least one neighbor colored j for any $j < i$. The result of the theorem then follows from the following lemma:

Lemma 1. *Let (G, c) be a circular colored graph with n vertices, and assume that c is a Grundy k-coloring of G. There is a constant K_0, such that $k \leq K_0 \log n$.*

The algorithm on planar graphs can be extended to the case of graphs with bounded genus, using only a small number depending on g of additional colors.

Theorem 3. *There is an online algorithm with bichromatic exchanges that can color any graph with genus g using $4g + O(\log \Delta)$ colors.*

This theorem is proved by looking at a generalization of the notion of circular drawing that we introduced for the planar case. The algorithm is exactly the same as in the planar case. The proof relies on a generalisation of Lemma 1 to this more general setting.

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