Genus and duality index of a hypermap

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Abstract

Topologically, a *hypermap* is a cellular embedding of a connected hypergraph into a closed connected surface. A more algebraic approach can also be followed. For instance, an oriented regular hypermap can be regarded as a triple $\mathcal{H} = (\Delta^+/H, x, y)$, with Δ^+ being the subgroup of index 2 in the triangle group Δ, and $\Delta^+/H = \langle x, y\rangle$ being the monodromy group of the oriented regular hypermap. The duality group of $\mathcal{H}$ is defined as the the minimal subgroup $D(\mathcal{H}) \trianglelefteq Mon(\mathcal{H})$ such that $\mathcal{H}/D(\mathcal{H})$ is a self-dual hypermap (a hypermap isomorphic to its dual). In this talk, we will try to show some relationships between the duality index (the order of $D(\mathcal{H})$) and the genus of a hypermap.

If G is the monodromy group of an oriented regular hypermap, then we can easily calculate its genus g using the formula:

$$-\mathcal{X} = 2g - 2 = |G| - V - E - F,$$

where V, E and F are the number of hypervertices, hyperedges and hyperfaces, respectively and $\mathcal{X}$ is the Euler characteristic. In an oriented regular hypermap of type (l, m, n) we have $|G| = mE = lV = nF$; therefore, the previous formula can be written in the following way:

$$-\mathcal{X} = 2g - 2 = |G|(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}).$$

Hence:

$$g = |G|(\frac{1}{2} - \frac{1}{2l} - \frac{1}{2m} - \frac{1}{2n}) + 1.$$

For each integer g there is a self dual oriented regular (hyper)map with genus g. It is well known that we can find a map of genus g of type $\{4g, 4g\}$ for every $g > 0$. Following a popular example (see, for instance, *Group Actions on Graphs, Maps and Surfaces*, a summary of a short course of lectures given by Marston Conder at the Group St Andrews conference in 2001), let G be the dihedral group of order $8g$ with generators u, v of order 2 and $4g$:

$$G =< u, v | u^2 = v^{4g} = (uv)^2 = 1 > .$$

If $r_0 = u$, $r_1 = uv$ and $r_2 = uv^{2g}$ we have: $r_0r_1 = v$ and $r_1r_2 = v^{2g-1}$ of order $4g$ and $r_2r_0 = v^{2g}$ of order 2. The map $\mathcal{M} = (G, r_0, r_1, r_2)$ is a g-sheeted covering of the torus map branched over its single vertex and its single face-center and is orientable since $\langle r_0r_1, r_1r_2\rangle = \langle v\rangle$ has index 2 in $G = D_{8g}$. The hypermap is chiral and its genus is equal to:

$$|D_{8g}|(\frac{1}{8} - \frac{1}{16g} - \frac{1}{16g}) + 1 = 8g(\frac{1}{8} - \frac{2}{16g}) + 1 = g.$$

$\mathcal{M}$ is then a map of type $\{4g, 4g\}$ and of genus g. The same can be said for the associated oriented hypermap:

$$\overline{\mathcal{M}} = (\overline{G}, r_or_1, r_1r_2) = (\langle v\rangle, v, v^{2g-1}).$$

This oriented regular map is self dual, since $(4g, 2g - 1) = 1$, and has also genus g.

If $g = 0$ (the sphere) the only oriented regular hypermaps are the Platonic solids and the two infinite families of types $(2, 2, n)$, $(n, 1, n)$ and their duals. Some of these are self-dual and others totally dual (if $D(\mathcal{H}) = Mon(\mathcal{H})$, or equivalently $H^{\Delta} = \Delta$, we say that the hypermap is *totally dual*) or with intermediate duality index.

The tetrahedron (with A_4 as monodromy group) is a self-dual map on the sphere. On the other hand, the cube and the octahedron (its dual) are examples of platonic solids that give rise to totally dual (hyper)maps of types $(3, 2, 4)$ and $(4, 2, 3)$, respectively. The same can be said about the icosahedron and dodecahedron, of types $(3, 2, 5)$ and $(5, 2, 3)$, respectively. These are just examples associated to the platonic solids but, on the sphere, it is possible to find an infinite number of totally dual hypermaps:

$$\mathcal{H} = (\mathbb{Z}_n = \langle t \rangle, t, 0).$$

These having a single hypervertex of valency n, n hyperfaces of valency 1 and one hyperedge of valency n. An infinite number of self-dual hypermaps on the sphere can also be obtained by using only dihedral groups as monodromy groups:

$$\mathcal{H} = (D_{2m}, x, y) \quad \text{with}$$

$$D_{2m} = \langle x, y | x^2, y^2, (xy)^m = 1 \rangle.$$

The same can be said for hypermaps of intermediate index (again, we have an infinite number of these) but a stronger statement can be made:

Theorem 1. *In the sphere we can find a non totally dual (hyper)map of each duality index d.*

A hypermap $\mathcal{H} = (G, x, y)$ of type (l, m, n) that has genus 1 (hypermap on the torus) must satisfy:

$$g = |G|(\frac{1}{2} - \frac{1}{2l} - \frac{1}{2m} - \frac{1}{2n}) + 1 = 1.$$

Which is the same as saying that:

$$\frac{1}{2} - \frac{1}{2l} - \frac{1}{2m} - \frac{1}{2n} = 0.$$

Therefore, on the torus, there are only three (infinite) families of oriented regular hypermaps (and their duals), whose types are $(2, 3, 6)$, $(2, 4, 4)$ and $(3, 3, 3)$.

A self-dual hypermap on the torus is $(\mathbb{Z}_4, 1, 1)$ of type $(4, 2, 4)$. Another example would be the hypermap $(A_3, (123), (123))$ of type $(3, 3, 3)$. If

$$G = \Delta(3, 3, 3) = \langle x, y | x^3 = y^3 = (xy)^3 = 1 \rangle$$

then the finite quotients of this group, by torsion-free normal subgroups, will give hypermaps of type $(3, 3, 3)$ and of genus 1. In fact, an infinite number of self-dual hypermaps can be constructed on the torus.

Totally dual hypermaps can also be found on the torus. For instance, $(\mathbb{Z}_4, 2, 1)$ is a totally dual hypermap of type $(2, 4, 4)$. Or we might take the group $G_n = \langle \alpha, \beta \rangle \leq S_{5+3n}$, for $n \in N$ with

$$\alpha = (1, 2),$$

$$\beta = (3,4,5)(6,7,8)...(3+3n, 4+3n, 5+3n).$$

Then $ord(\alpha) = 2$, $ord(\beta) = 3$ and $ord(\alpha\beta) = 6$. It follows that the hypermap $\mathcal{H} = (G_n, \alpha, \beta)$ is totally dual of type $(2,6,3)$. We can easily obtain other totally dual hypermaps on the torus by taking finite quotients, by torsion-free normal subgroups, of the universal hypermap of type $(2,6,3)$.

However, not all hypermaps on the torus are self-dual or totally dual. Examples of hypermaps with intermediate duality index can also be found: $(\mathbb{Z}_6, 1, 3)$ of type $(6,2,3)$ has duality index 3.

For genus 2, one example of a totally dual hypermap is the hypermap of type $(3,3,4)$, described in [3], with $SL_2(3)$ as monodromy group (and that is obviously totally dual since 3 and 4 are co-prime). Another one is the map of type $\{3,8\}$ listed by Conder (http://www.math.auckland.ac.nz/conder/).

For every $g \in \mathbb{N}$ we can get an infinite number of totally dual hypermaps of genus greater than g by using, for instance, Hurwitz maps (hypermaps of type $(3,2,7)$). Higman proved that almost all alternating groups are quotients of the triangle group $\Delta(2,3,7)$. Because the genus is given by

$$\frac{1}{84}|A_n| + 1 = \frac{n!}{168} + 1,$$

we can make it as big as we want. Some other examples of totally dual Hurwitz maps can be obtained using Luchini's result in [2] but in this case with $SL_n(q)$ as monodromy group. Since explicit generators are given in [2], not only can we obtain a sequence of hypermaps of growing genus but also the description of those hypermaps.

We have already proved that we can always find a self-dual hypermap for every genus. But we can also find, for each integer $g > 1$, a non self-dual hypermap of that genus g. In fact, we can do this using only hypermaps of a certain type:

Theorem 2. *For each integer $g > 1$ there is a non self-dual (and non chiral) hypermap of genus g with vertices, edges and faces of the same order.*

References

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[3] António Breda D'Azevedo, Gareth Jones, Rotary Hypermaps of Genus 2, Contributions to Algebra and Geometry, Vol.42 (2001), No.1, 39-58