Infinite n-ordered Graphs and Independent Distinguishing Sets

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Abstract

A graph G is n-ordered if its vertices can be enumerated so each vertex has no more than n neighbours appearing earlier in the enumeration. Consider the following process for generating an infinite n-ordered graph. Let G_0 be a finite n-ordered graph. Form G_{i+1} from G_i as follows: For each $S \subseteq V(G_i)$ of cardinality at most n, add a new vertex adjacent to every vertex in S. Taking the limit of this process gives a n-ordered infinite graph with countably many vertices. This construction method is reminiscent of the deterministic construction of the Rado graph. In this work we use universal adjacency properties to study the class of all infinite graphs that admit such a construction. For these graphs, we provide a probabilistic construction and also show that these universal adjacency properties imply the existence of independent distinguishing sets.

Unless otherwise stated, all graphs are assumed to be simple. For a graph G and a pair of disjoint of subsets $A, B \subseteq V(G)$ a vertex of $V(G) \setminus (A \cup B)$ is *correctly joined to A and B* if it adjacent to every vertex in A and no vertex in B. A graph is *n-existentially closed* (n-e.c.) if for every pair of disjoint subsets $A, B \subseteq V(G)$ so that $|A \cup B| \leq n$ there is a vertex correctly joined to A and B. Though for any fixed n, nearly every graph is n-e.c., constructing explicit examples of n-e.c. graphs and graph families is a non-trivial task utilizing a wide variety of combinatorial tools [1]. The *Rado graph*, denoted R, is the graph with a countably infinite set of vertices, where each pair of vertices is adjacent with some fixed probability $p \in (0, 1)$. Surprisingly for any $p \in (0, 1)$, the graph resulting from this process is the same [4]. The infinite graph R is characterized as being the unique infinite graph that is n-e.c. for all n. Using this characterization, a number of other construction methods have been shown to generate R. We are interested in the following deterministic construction: Let $G_0 = K_1$. In step $t = i + 1$, form G_{i+1} from G_i as follows. For each $S \subseteq V(G_i)$ add a vertex adjacent to every vertex in S. The infinite graph formed as $t \to \infty$ is n-e.c. for all $n \geq 1$, and thus is isomorphic to R. We direct the reader to a comprehensive resource on various aspects of R [4]. In this work we study graphs formed by the process above when we vary the input graph, G_0, and put an upper bound on the cardinality of S.

In [2] the authors consider a variation of the deterministic construction of R. Let n be a positive integer. Let $G_0 = K_n$. In step $t = i+1$, form G_{i+1} from G_i as follows. For each $S \subseteq V(G_i)$ so that $|S| = n$ add a vertex adjacent to every vertex in S. The graph formed by considering the limit as $t \to \infty$ is denoted $R^{(n)}$. Following this, we denote by $R^{(\leq n)}$ the graph formed analogously to $R^{(n)}$ by considering in each step $S \subseteq V(G_i)$ so that $|S| \leq n$.

The infinite graphs R, $R^{(n)}$ and $R^{(\leq n)}$ satisfy a stronger existential closure property: A graph G is *strongly n-existentially closed* when for every pair of finite disjoint subsets $A, B \subseteq V(G)$ so that $|A| \leq n$, there is a vertex correctly joined to A and B. That there is no restriction on the cardinality of B implies for every n that every strongly n-existentially closed graph is infinite.

Following [2], for fixed n, G is *n-ordered* if there exists a well-ordering of its vertices $(x_i : i \in I)$, where I is finite or I has the order-type $\mathbb{N}$, so that each x_j has at most n neighbours x_i with $i < j$.

Theorem 1. *Let n be a fixed positive integer. Let G be a finite n-ordered graph.*

1. *Every induced subgraph of $R^{(n)}$ and $R^{(\leq n)}$ is n-ordered.*

2. *There are countably infinitely many induced subgraphs of $R^{(n)}$ isomorphic to G.*

3. *There are countably infinitely many induced subgraphs of $R^{(\leq n)}$ isomorphic to G.*

Results for $R^{(n)}$ in Theorem 1 appear in [2]. We include them here for completeness. Theorem 1 suggests, perhaps, that $R^{(n)}$ and $R^{(\leq n)}$ are isomorphic. This is not the case.

Theorem 2. *Let n_1 and n_2 be positive integers. $R^{(n_1)} \ncong R^{(\leq n_2)}$.*

In the deterministic construction of R given above, letting $G_0 = G$ for any finite graph G yields a graph isomorphic to R. However, in the construction of $R^{(n)}$ and $R^{(\leq n)}$ this is not the case. Consider, for example, the infinite graph constructed as $R^{(\leq n)}$ by letting $G_0 = K_{n+2}$. The resulting infinite graph contains K_{n+2} as an induced subgraph. By Theorem 1 $R^{(\leq n)}$ has no subgraph isomorphic to K_{n+2}.

Let H be a finite graph. Let $R^{(H,\leq n)}$ be the infinite graph generated as $R^{(\leq n)}$ by letting $G_0 = H$. Analogously define $R^{(H,n)}$. In [2] the authors introduce $R^{(H,n)}$. They leave open the question of determining when $R^{(H,n)} \cong R^{(H',n)}$ for $H \ncong H'$. We answer this question in Lemma 1.

It is easily checked that both $R^{(H,\leq n)}$ and $R^{(H,n)}$ are strongly n'-e.c., for all $1 \leq n' \leq n$, but not strongly n'-e.c. for any $n' > n$. Parts 2 and 3 of Theorem 1 hold for $R^{(H,n)}$ and $R^{(H,\leq n)}$. Part 1 of Theorem 1 holds for $R^{(H,n)}$ and $R^{(H,\leq n)}$ if and only if H is n-ordered.

For the remainder of this work, we are interested in infinite graphs that are members of the following families:

$$\Gamma_n = \{R^{(H,n)} \mid H \text{ is a finite graph and } |V(H)| \geq n\}$$

$$\Gamma_{\leq n} = \{R^{(H,\leq n)} \mid H \text{ is a finite graph}\}$$

These two families are not only distinct, but also non-intersecting.

Theorem 3. *Let n_1 and n_2 be fixed positive integers. A countably infinite graph G is contained in at most one of Γ_{n_1} and $\Gamma_{\leq n_2}$.*

For a pair of graphs H and H' we write $H \prec_n H'$ when H' can be formed from H by iteratively adding vertices of degree n. We write $H \prec_{\leq n} H'$ when H' can be formed from H by iteratively adding vertices of degree at most n. By definition we have $H \prec_n R^{(H,n)}$ and $H \prec_{\leq n} R^{(H,\leq n)}$.

Lemma 1. *Let G and H be finite graphs.*

1. *$R^{(G,n)} \cong R^{(H,n)}$ if and only if there exists a finite graph Z such that $G \prec_n Z$ and $H \prec_n Z$.*

2. *$R^{(G,\leq n)} \cong R^{(H,\leq n)}$ if and only if there exists a finite graph Z such that $G \prec_{\leq n} Z$ and $H \prec_{\leq n} Z$.*

Lemma 1 re-frames the question of isomorphism of a pair of infinite graphs as a question regarding properties of finite graphs. For Γ_n, necessary and sufficient conditions for the existence of such a finite graph Z remain elusive. Further, it is not immediately clear that the associated decision problem is a member of NP. For $\Gamma_{\leq n}$ such necessary and sufficient conditions are given in Theorem 4.

For a graph G, the *k-core* of G is the maximum induced subgraph of minimum degree k contained in G. If G has no such subgraph, then the *k-core* of G is defined to be K_1. In this case

we say that G has a *trivial k-core*. The k-core of a graph is unique up to isomorphism [7]. Observe that if H_0 is the smallest induced subgraph of H such that $H_0 \prec_{\leq n} H$, then H_0 is the $n+1$-core of H. Thus H is n-ordered if and only if the $n+1$-core of H is trivial. The concept of k-core plays an important role in the study of $\Gamma_{\leq n}$.

Theorem 4. *$R^{(G,\leq n)} \cong R^{(H,\leq n)}$ if and only if G and H have the same $n+1$-core.*

Corollary 1. *For fixed $n \geq 1$, there is a one-to-one correspondence between $n+1$-cores and elements of $\Gamma_{\leq n}$.*

The k-core of a finite graph can be computed efficiently by iteratively removing vertices of degree at most $k-1$ [7]. As such for fixed $n \geq 1$, given G and H, the decision problem of determining if $R^{(G,\leq n)} \cong R^{(H,\leq n)}$ is contained in P.

Graphs contained in Γ_n or $\Gamma_{\leq n}$ admit a deterministic construction reminiscent of that of the Rado Graph. And so it is natural to wonder whether graphs contained in these families also admit a probabilistic construction.

Consider the following process, which we name *Model* ($\leq n$). Let n be a fixed positive integer. Let H be a non-trivial finite graph with vertex set $u_1^0, u_2^0, \ldots u_{|V_H|}^0$. Let $H_0 = H$. For fixed $t > 0$ let $n(t) = \min\{n, |V(H_{t-1})|\}$. We form H_t from H_{t-1} by adding new vertices $\{x_1^t, x_2^t, \ldots, x_{n(t)}^t\}$ so that x_i^t is adjacent to exactly i vertices of H_{t-1}. Let $N_{H_t}(x_i^t)$ denote the neighbour set of x_i^t in H_t. For fixed i, and each $S = \{u_{j_1}, u_{j_2}, \ldots u_{j_i}\}$, let $\mu(S) = 2^{-(j_1+j_2+\cdots+j_i)}$. Let

$$C_i^t = \sum_{S \subset V(H_{t-1}), |S|=i} \mu(S)$$

and let

$$\mathbb{P}(N_{H_t}(x_i^t) = S) = \frac{\mu(S)}{C_i^t}.$$

Model (n), defined analogously, first appears in [2] with the restriction $H_0 = K_n$. Here we lift this restriction and require only that H_0 has at least n vertices.

Theorem 5. *Let $G = \lim_{t\to\infty} H_t$, where H_t was generated with Model ($\leq n$). With probability 1, $G \cong R^{(H_0,\leq n)}$.*

Theorem 6. *Let $G = \lim_{t\to\infty} H_t$, where H_t was generated with Model (n). With probability 1, $G \cong R^{(H_0,n)}$.*

We close our discussion with an application of the strong 1-e.c. property to the study of graph distinguishing. Since every infinite graph of the form $R^{(H,n)}$ or $R^{(H,\leq n)}$ is strongly 1-e.c., our results allow us to compute the distinguishing number of any infinite graph contained in either Γ_n or $\Gamma_{\leq n}$ for any $n \geq 1$.

Let G be a graph and $c : V(G) \to \{1, 2, 3, \ldots, k\}$. Let $Aut_c(G)$ be the set of automorphisms f of G with the property that for all $v \in V(G)$ we have $c(f(v)) = c(v)$. That is, $Aut_c(G)$ is the set of automorphisms of G that preserve c. Let ϵ denote the trivial automorphism. If $Aut_c(G) = \epsilon$, we say that c is *k-distinguishing*. Informally, c "breaks" all non-trivial automorphisms of G.

Distinguishing may be re-framed in the language of graph homomorphism. We say a homomorphism $\phi : G \to H$ is *distinguishing* when $Aut_\phi(G) = \epsilon$. If H is a reflexive complete graph on k vertices, then a distinguishing homomorphism to H is exactly a k-distinguishing function.

For a graph G, the *distinguishing number of* G, denoted $D(G)$, is the least integer k such that there exists a distinguishing homomorphism of G to a reflexive target on k vertices. If we restrict the target of such a homomorphism to be an irreflexive graph, then we arrive at the *distinguishing chromatic number* of G, denoted $D_\chi(G)$. See [3] and [6] for more on graph distinguishing and the distinguishing chromatic number.

Let $D_i(G)$ be the *independent distinguishing number* of G. That is, $D_i(G)$ is the least k such that there exists a distinguishing homomorphism to a target H on k vertices so that H is not reflexive. We have the following inequalities for all graphs on $n \geq 2$ vertices whose automorphism groups are non-trivial.

$$n \geq D_\chi(G) \geq \chi(G) \geq D_i(G) \geq D(G) \geq 2 \tag{1}$$

When $D_i(G) = 2$, G contains an independent set I so that the function:

$$c(v) = \begin{cases} 1 & v \in I \\ 2 & otherwise \end{cases}$$

is 2-distinguishing. Call such an independent set an *independent distinguishing set.*

Theorem 7. *If G is strongly 1-e.c. and has countably many vertices, then G has an independent distinguishing set.*

Corollary 2. *The Rado graph contains an independent distinguishing set.*

Corollary 3. *For all $n \geq 1$, if $G \in \Gamma_n$ or $G \in \Gamma_{\leq n}$, then G contains an independent distinguishing set.*

Corollary 3 together with Equation 1 imply that any graph G for which $D(G) > 2$ is not a member of Γ_n or $\Gamma_{\leq n}$. As the distinguishing number of infinite graphs is well studied (for example, see [5]), the distinguishing number of an infinite graph G can be used to determine if it is a candidate element of Γ_n or $\Gamma_{\leq n}$.

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