Total coloring of graphs with minimum sum of colors; existence of T-strong graphs and trees.

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Abstract

The total chromatic sum of a graph is the minimum sum of colors (natural numbers) taken over all proper colorings of vertices and edges of a graph. We provide infinite families of graphs for which the minimum number of colors to achieve the total chromatic sum is larger than the total chromatic number and discuss two conjectures about total colorings of graphs and trees.

1 Introduction

Consider a proper coloring ϕ of vertices of a graph G using natural numbers; i.e. $\phi : V(G) \to N$ and $\phi(u) \neq \phi(v)$ whenever uv is an edge of G. The *chromatic sum* of G, denoted $\Sigma(G)$, is the minimum sum $\sum_{v \in V(G)} \phi(v)$ taken over all proper colorings ϕ of G. A coloring is *optimal* if the sum of colors equals $\Sigma(G)$.

This idea was introduced by Kubicka [4] in 1989, and since then much more work has been done with calculating the chromatic sums of graphs, generating algorithms to find chromatic sums and optimal colorings, and calculating the complexity of finding chromatic sums of graphs in certain families. Erdös, Kubicka, and Schwenk [2] constructed infinite families of graphs for which the minimum number of colors necessary to get an optimal coloring of G was larger than $\chi(G)$. This graph parameter, the minimum number of colors necessary for an optimal coloring, is called the *strength* of G, and denoted by $\sigma(G)$ [3]. In [2], it is shown that even trees can have arbitrarily high strength, even though their chromatic number is 2. In fact, Erdös, Kubicka, and Schwenk [2] found for every $k \geq 3$ the smallest tree of strength k. In [3], Jiang and West also constructed trees with strength k but not of minimum order but of minimum maximum degree, $\Delta = 2k - 2$.

We say that a graph G is *strong* if $\chi(G) < \sigma(G)$. The smallest strong graph is the tree on eight vertices given in Figure 1.



Figure 1: The smallest strong tree.

These color-sum concepts can be applied to edge coloring as well. In an analogous way, one can define the *edge chromatic sum* of a graph, its edge strength σ', and ask the question of whether or not $\chi' = \sigma'$. In 1997, Mitchem, Morris, and Schmeichel [7] proved that every graph has a proper edge coloring with minimum sum that uses only Δ or $\Delta + 1$ colors. This implies that the only way for a graph to have $\chi' < \sigma'$ is to have both $\chi' = \Delta$ and $\sigma' = \Delta + 1$. We say that a graph G with this property, namely $\chi'(G) < \sigma'(G)$, is *E-strong*. The smallest known E-strong graph M is presented in Figure 2.

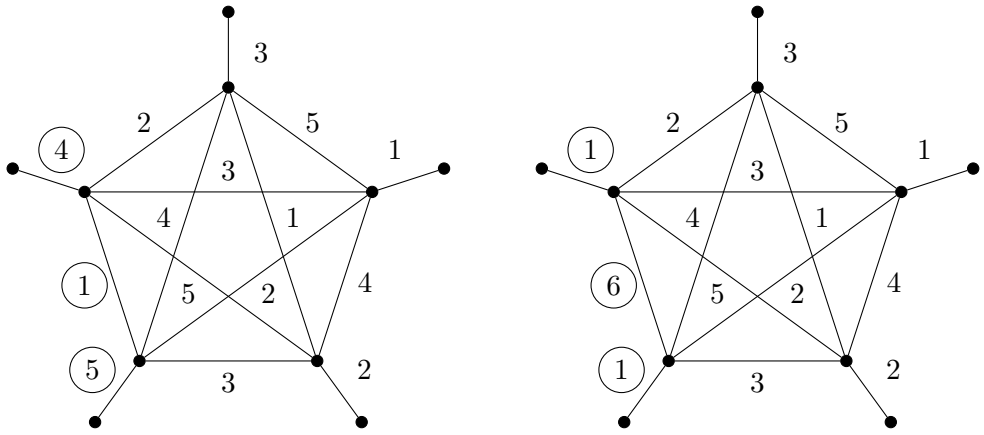


Figure 2: E-strong graph M with $\Delta = 5$, $\chi' = 5$, and $\sigma' = 6$. On the left side with five colors used, the sum of the colors is 45. If we introduce a sixth color and change the colors of the edges whose color labels are circled, we obtain a coloring with sum 43.

In the same paper, Mitchem et al [7] provide two other infinite families of E-strong graphs. Graphs from one of these families are "almost" regular i.e. $\Delta(G) = \delta(G) + 1$.

We define similar concepts for total colorings of graphs in the next section. In the last part we present several infinite families of graphs for which we need more colors for optimal total coloring than the total chromatic number. The existence of T-strong trees is discussed at the end.

2 Total Colorings and Total Chromatic Sum of Graphs

A *total coloring* ϕ of G is a mapping $\phi : V(G) \cup E(G) \to N$. A total coloring of G is *proper* if no pair of adjacent or incident elements (vertices or edges) is assigned the same color. A total-k-coloring ϕ of G is a proper total coloring that uses k colors. The *total chromatic number* $\chi^T(G)$ of a graph G is the smallest number k for which G has a total-k-coloring. The famous Total Coloring Conjecture stating that

$$\chi^T(G) \leq \Delta(G) + 2$$

for every graph G, where $\Delta(G)$ is the maximum degree of G, was posed independently by Vizing [8, 9] and Mehdi Behzad [1].

We focus our attention to the concept of a total chromatic sum. Let us denote by $\Phi(G)$ the set of all total colorings of G. For every graph G, define the function $sum : \Phi(G) \to \mathbb{N}$ to return the sum of colors of all the elements of G,

$$sum(\phi) = \sum_{x \in V(G) \cup E(G)} \phi(x).$$

The *total chromatic sum* $\Sigma^T(G)$ of a graph G is defined as the smallest value of $sum(\phi)$ among all colorings $\phi \in \Phi(G)$,

$$\Sigma^T(G) = \min_{\phi \in \Phi(G)} sum(\phi).$$

As for other graph coloring concepts, we will call a total coloring ϕ of G *optimal* if $sum(\phi) = \Sigma^T(G)$. For each total coloring ϕ, define $\sigma^T(\phi)$ to be the number of colors used in ϕ; then the *total strength*

$\sigma^T(G)$ of G is defined as the smallest value of $\sigma^T(\phi)$ among all optimal total colorings ϕ of G,

$$\sigma^T(G) = \min\{\sigma^T(\phi) : \phi \in \Phi(G) \text{ and } sum(\phi) = \Sigma^T(G)\}.$$

We say that a graph G is *T-strong* if $\sigma^T(G) > \chi^T(G)$. Total chromatic sum and the related parameters were introduced by Leidner [6], where the total chromatic sum was determined for many families of graphs including: complete graphs, paths, cycles, complete bipartite graphs, full binary trees, and hypercubes Q_n for $n \leq 6$.

3 Existance of T-strong graphs and trees

The smallest T-strong graph is a 6-cycle with a diametral chord, or equivalently the graph $P_2 \times P_3$. Two colorings, the first using $\chi^T(G)$ colors and the second that is optimal and uses one more color are depicted in Figure 3. Leidner [6] verified by exhaustive computer search that $P_2 \times P_3$ is the smallest T-strong graph and the only one of order smaller than 9.

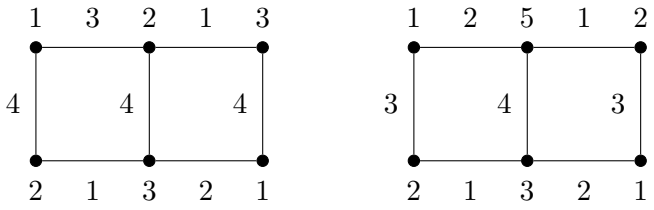


Figure 3: The grid graph $P_2 \times P_3$ total-colored in two ways.

In [6] and [5], several infinite families of T-strong graphs were constructed. Each graph from the following list is T-strong:
(1) Cycles of length $3n$, $n \geq 2$, with one chord joining vertices at distance congruent to 3 along the cycle. Those graphs have $\Delta = 3, \chi^T = 4$, and $\sigma^T = 5$.
(2) Cycles of length $3n$, $n \geq 2$, with two independent chords with proper distances along the cycle. For those graphs also $\Delta = 3, \chi^T = 4$, and $\sigma^T = 5$.
(3) Graphs G obtained from M, the graph from Figure 2, by attaching a copy of K_{2k+1} to each vertex. Here $\Delta(G) = 2k + 5, \chi^T(G) = 2k + 6$, and $\sigma^T(G) = 2k + 7$.

Two conjectures about the total strength of a graphs were stated in [5].
Conjecture 1. *For every graph G,* $\chi^T(G) \leq \sigma^T(G) \leq \chi^T(G) + 1$.
Conjecture 2. *No tree is T-strong.*

We verify that Conjecture 1 is true for trees, sharply contrasting the situation for vertex coloring in which the strength of a tree can be arbitrarily higher than its chromatic number.

No example of a T-strong tree is known, but we were able to verify that Conjecture 2 is true for several families of trees. We call a tree $\Delta\Delta$-*free* if it does not contain two adjacent vertices of maximum degree.

Conjecture 2 is true, it means that $\chi^T(G) = \sigma^T(G)$, for every tree from the following families:
(1) Full binary trees.
(2) Caterpillars.
(3) Trees with diameter bounded by 4.
(4) $\Delta\Delta$-free trees.

In addition, if there exists a T-strong tree G, then the structure of the tree and an optimal total coloring of G must be as follows:
(1) G must have two adjacent vertices, say u and v, both of degree $\Delta(G)$.
(2) Color $\Delta + 2$ must occur at one of those vertices, say v.
(3) The three palettes of colors occurring on vertices adjacent to v (not counting u), the edges incident to v (not counting uv), and the edges incident to u (without uv as well) must be identical.

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