Characterization of forbidden subgraphs for bounded star chromatic number

Ilkyoo Choi — **Hankuk University of Foreign Studies**
Ringi Kim — **KAIST**
Boram Park — **Ajou University**

Abstract

A proper coloring of a graph is a partition of its vertex set into independent sets. In 1973, Grünbaum suggested the concept of proper colorings that also forbid bicolored subgraphs, but no specific name was given. We revive this notion by defining an *H-avoiding coloring* to be a proper coloring that forbids a bicolored subgraph H. Examples are 2-distance coloring and star coloring, which forbids a bicolored P_3 and P_4, respectively.

When considering the class $\mathcal{C}$ of graphs with no F-induced subgraph, it is known that graphs in $\mathcal{C}$ have bounded chromatic number if and only if F is a subgraph of K_2. We study this phenomena for the class of graphs with no F-subgraph for H-avoiding coloring. We completely characterize all graphs F for a large class of graphs H. As corollaries, we characterize all graphs F where the class of graphs with no F-subgraph has bounded star chromatic number and also acyclic chromatic number.

Given a graph G, a *proper k-coloring* of G is a partition of its vertex set $V(G)$ into k parts such that there is no edge with both endpoints in the same part. In other words, there exists a function φ on $V(G)$ to a set of k colors such that $\varphi(x) \neq \varphi(y)$ for each edge xy of G. The *chromatic number* of G, denoted $\chi(G)$, is the minimum k such that G has a proper k-coloring.

Since a proper coloring of a graph G is a partition of its vertex set, there is an intimate relation between the structure of G and its chromatic number $\chi(G)$. Since a complete graph requires all of its vertices to receive different colors, the size of a largest clique of a graph G, the clique number of G, is a trivial lower bound on $\chi(G)$. There was great interest in characterizing graphs where the aforementioned trivial lower bound is also an upper bound for the chromatic number; graphs in which the clique number is equal to the chromatic number for all induced subgraphs are also known as *perfect graphs*. Conjectured by Berge [1] in 1961, the characterization of perfect graphs in terms of forbidden induced subgraphs was completed by Chudnovsky et al. [2]. It is now known as the Strong Perfect Graph Theorem, which states that a graph is a perfect graph if and only if it contains neither an odd cycle nor a complement of an odd cycle as an induced subgraph.

As a generalization of perfect graphs, researchers started investigating which classes of graphs with a forbidden structure are *χ-bounded*; a class $\mathcal{C}$ of graphs is χ-bounded if there exists a function such that the chromatic number of each graph G in $\mathcal{C}$ is bounded by the same function of the clique number of G. By the random construction of Erdős [5], for each k and g, we know that there exists a graph G with $\chi(G) \geq k$ and minimum cycle length at least g. This implies that for the class of graphs without F as an induced subgraph to be χ-bounded, it is necessary that F is a forest. Gyárfás [7] and Sumner [15] independently conjectured that the necessary condition is also sufficient, namely, for a forest F and an integer q, there is a function $f(F, q)$ such that the chromatic number of a graph that contains neither F as an induced subgraph nor a clique of size q is bounded by $f(F, q)$. This conjecture is known to be true for paths [8], stars, and some additional forests [9, 11, 12, 14, 4], but is mostly wide open. We redirect the authors to [3, 13] for more information regarding recent progress on various conjectures regarding χ-bounded graphs.

A natural question to ask is changing the above question from forbidding two structures (a graph F and a clique of size q) to a single structure. When the structure is defined to be an

induced subgraph, since F must be both a complete graph and a forest, we obtain the following fact:

Fact 1. *The class of graphs without F as an induced subgraph is χ-bounded if and only if F is a complete graph on at most two vertices.*

We are interested in when the forbidden structure is a subgraph, which is not necessarily induced. We define an *F-free graph* to be a graph that does not contain F as a subgraph. As in the case of induced subgraphs, in order for the class of F-free graphs to be χ-bounded, it must be the case that F is a forest by the previously mentioned construction by Erdős. However, since a graph with minimum degree at least $k-1$ contains all trees of order k, we know that for a tree F, an F-free graph must contain a vertex of degree at most $k-2$. This further implies that G is $(k-2)$-degenerate, and an ordering of the vertices exists such that each vertex has at most $k-2$ later neighbors. Since we can greedily color the vertices in the reverse order with $k-1$ colors, we know that G is $(k-1)$-colorable. Therefore, we conclude the following:

Fact 2. *The class of F-free graphs is χ-bounded if and only if F is a forest.*

0.1 Our contributions

In this paper, we are interested in obtaining characterizations similar to Fact 1 for various other coloring parameters. Instead of dealing with each coloring parameter separately, we define a generalized coloring parameter that captures the constraints of several well-known coloring parameters in the literature such as 2-distance coloring, acyclic coloring, and star coloring. All these coloring parameters are proper colorings that also forbid bicolored subgraphs; in addition to being a proper coloring, 2-distance coloring, acyclic coloring, and star coloring forbid a bicolored path on three vertices, bicolored cycles, and a bicolored path on four vertices, respectively. This notion was suggested by Grünbaum [6] in 1973, but no specific name was given. We revive this notion by defining an *H-avoiding coloring* to be a proper coloring that forbids a bicolored subgraph H. We direct the readers to the well-known graph coloring book [10] of Jensen and Toft for more information regarding various coloring parameters.

We say a class $\mathcal{C}$ of graphs has bounded H-avoiding chromatic number if there is a function $f(H)$ such that each graph in $\mathcal{C}$ has an H-avoiding $f(H)$-coloring. The purpose of this paper is to investigate the phenomena in Fact 1 and 2 for H-avoiding coloring. Namely, we initiate the study of characterizing all graphs F for a given H where the class of F-free graphs has bounded H-avoiding chromatic number. Note that if H is not 2-colorable, then H-avoiding coloring is the same as proper coloring. We start with the following result simple result regarding stars:

Theorem 1. *Let H be a star. The class of F-free graphs has bounded H-avoiding chromatic number if and only if F is a star.*

A *d^+-vertex* is a vertex of degree at least d. Our main result is the following theorem that deals with a much larger class of bipartite graphs:

Theorem 2. *Let H be a connected bipartite graph that is not a star where all vertices in one part have degree at most 2. The class of F-free graphs has bounded H-avoiding chromatic number if and only if F is a forest where each pair of 3^+-vertices has even distance.*

The following corollary is a direct consequence of our main result:

Corollary 1. *For a graph F, the class of F-free graphs has bounded star chromatic number if and only if F is a forest where each pair of 3^+-vertices has even distance.*

The following result is not a direct consequence of Theorem 2, nonetheless it follows from our lemmas.

Corollary 2. *For a graph F, the class of F-free graphs has bounded acyclic chromatic number if and only if F is a forest where each pair of two 3^+-vertices has even distance.*

References

[1] C. Berge. Fäbung von graphen, deren sämtliche bzw. deren ungerade kreise starr sind. *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe*, 10:114, 1961.

[2] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. *Ann. of Math. (2)*, 164(1):51–229, 2006.

[3] M. Chudnovsky, A. Scott, and P. Seymour. Induced subgraphs of graphs with large chromatic number. II. Three steps towards Gyárfás' conjectures. *J. Combin. Theory Ser. B*, 118:109–128, 2016.

[4] M. Chudnovsky, A. Scott, and P. Seymour. Induced subgraphs of graphs with large chromatic number. xii. distant stars, 2017.

[5] P. Erdős. Graph theory and probability. *Canad. J. Math.*, 11:34–38, 1959.

[6] B. Grünbaum. Acyclic colorings of planar graphs. *Israel J. Math.*, 14:390–408, 1973.

[7] A. Gyárfás. On Ramsey covering-numbers. pages 801–816. Colloq. Math. Soc. Janós Bolyai, Vol. 10, 1975.

[8] A. Gyárfás. Problems from the world surrounding perfect graphs. *Tanulmányok—MTA Számitástech. Automat. Kutató Int. Budapest*, (177):53, 1985.

[9] A. Gyárfás, E. Szemerédi, and Zs. Tuza. Induced subtrees in graphs of large chromatic number. *Discrete Math.*, 30(3):235–244, 1980.

[10] T. R. Jensen and B. Toft. *Graph coloring problems.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New York, 1995. A Wiley-Interscience Publication.

[11] H. A. Kierstead and S. G. Penrice. Radius two trees specify χ-bounded classes. *J. Graph Theory*, 18(2):119–129, 1994.

[12] H. A. Kierstead and Y. Zhu. Radius three trees in graphs with large chromatic number. *SIAM J. Discrete Math.*, 17(4):571–581, 2004.

[13] A. Scott and P. Seymour. Induced subgraphs of graphs with large chromatic number. I. Odd holes. *J. Combin. Theory Ser. B*, 121:68–84, 2016.

[14] A. D. Scott. Induced trees in graphs of large chromatic number. *J. Graph Theory*, 24(4):297–311, 1997.

[15] D. P. Sumner. Subtrees of a graph and the chromatic number. In *The theory and applications of graphs (Kalamazoo, Mich., 1980)*, pages 557–576. Wiley, New York, 1981.