On Min-max Pair in Tournaments

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Abstract

We prove the following result in this paper.

Theorem: Let $T = (V(T), E(T))$ be a tournament of order $n \geq 3$, x a vertex of maximum out-degree and y a vertex of minimum out-degree of T. If $yx \in E(T)$ then there exists a path of length i from x to y for any i with $2 \leq i < n$; and if $xy \in E(T)$, then there exists a path of length i from x to y for any i with $3 \leq i < n$ unless T is in an exception class.

From this we obtain the following well-known results.

Corollary 1: (Alspach [1]) If e is an arc of a regular tournament of order $n \geq 3$, then e is contained in cycles of all lengths m with $3 \leq m \leq n$.

Corollary 2: (Alspach et al [2]) If $e = xy$ is an arc of a regular tournament of order $n \geq 7$, then there exists a path from x to y of length m for all m with $3 \leq m < n$.

A *tournament* $T = (V(T), E(T))$ *of order* n is an orientation of a complete graph of order n, with vertex set $V(T)$ and arc set $E(T)$. Undefined terminology can be found in [3].

If uv is an arc of T, then we say that u *dominates* v in T. The number of vertices dominated by u is called the *out-degree* of u and is denoted by $d^+(u)$. The number of vertices dominating u is called the *in-degree* of u and is denoted by $d^-(u)$.

For disjoint vertex sets A and B, we use $A \Rightarrow B$ to denote that every vertex in A dominates every vertex in B. If $A = \{a\}$ then we also use $a \Rightarrow B$ to denote $A \Rightarrow B$. A pair of distinct vertices x, y of T is called a *min-max pair* if one is a vertex of minimum out-degree and the other is

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of maximum out-degree. (Note that a vertex of minimum out-degree is a vertex of maximum in-degree.) Let xy be an arc for a min-max pair x, y. It is called a *min-max arc* if x has minimum out-degree, and a *max-min arc* otherwise. Given two distinct vertices u and v it is interesting to know if there is a path of length i from u to v. Our first result is:

Theorem 1. *Let T be a tournament of order $n \geq 3$. If yx is a min-max arc, then there exists a path of length i from x to y for any i with $2 \leq i < n$.*

Corollary 2. *(Alspach [1]) If e is an arc of a regular tournament of order $n \geq 3$, then e is contained in cycles of all lengths m with $3 \leq m \leq n$.*

Let $N^+(u) = \{v|uv \in E(T)\}$ and $N^-(u) = \{v|vu \in E(T)\}$. For two disjoint vertex sets X and Y of T, let $[X, Y] = \{uv \in E(T)|u \in X, v \in Y\}$.

If xy is a max-min arc, then it is not guaranteed to have a hamiltonian path from x to y. A simple example is the tournament on $\{x, y, z\}$ with arcs xy, yz and zx. But we can prove that, excluding a few special cases, there does exist a hamiltonian path from x to y for max-min arc xy.

For convenience we assume that x, y is a min-max pair with x having maximum out-degree. We define vertex sets A, B, C and D as: $A = N^+(x) \cap N^-(y)$, $B = N^+(x) \cap N^+(y)$, $C = N^-(x) \cap N^-(y)$ and $D = N^-(x) \cap N^+(y)$. Notice that $V(T) = A \cup B \cup C \cup D \cup \{x, y\}$.

If $D = \{d\}, B = \emptyset$ and $A \cup C \Rightarrow d$, or $D = \{d\}, C = \emptyset$ and $d \Rightarrow A \cup B$, or $B = C = \emptyset$, $D = \{d_1, d_2\}$, $A = \{a\} \cup A_0$ with $d_1 \Rightarrow A_0$, $A_0 \Rightarrow d_2$ and arcs $\{ad_1, d_1d_2, d_2a\}$, where $0 \leq |A_0| \leq 1$, then we say that T is in an *exception class*.

If T is a tournament in the exception class, then there is no hamiltonian path from x to y. We prove this as follows. Notice that in the first case of the exception class any path from d to y must use arc dx. If there is a hamiltonian path from x to y, then there must be a path from d to y without using dx, a contradiction. Similarly, in the second case of the exception class any path from x to d must use arc yd. If there is a hamiltonian path from x to y then there must be a path from x to d without using yd, also a contradiction. The third case of the exception class can be checked directly.

Our next result is:

Theorem 3. *Let T be a tournament T of order $n \geq 3$. If xy is a max-min arc, then there exists a path of length i from x to y for any i with $3 \leq i < n$ unless T belongs to the exception class.*

Corollary 4. *(Alspach et al [2]) If $e = xy$ is an arc of a regular tournament of order $n \geq 7$, then there exists a path from x to y of length m for all m with $3 \leq m < n$.*

In a directed graph G, a vertex z is said to *satisfy the $x \to y$ path condition* (or just path condition for short, if x and y are known from the context.) if there exists a path from x to z and also a path from z to y. We call G an *$x \to y$ reachable graph* (or just *reachable graph* for short.) if it is acyclic and every vertex z of G satisfies the $x \to y$ path condition.

We need the following result of Thomassen [4] in our proofs:

Theorem 5. *(Theorem 5.2 in [4]) Let T be a tournament and u, v distinct vertices of T. Then T has a hamiltonian path from u to v if and only if T contains a spanning $u \to v$ reachable graph.*

References

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