Edge correlations in random regular hypergraphs and applications to subgraph testing

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Abstract

Compared to the classical binomial random (hyper)graph model, the study of random regular hypergraphs is made more challenging due to correlations between the occurrence of different edges. We develop an edge-switching technique for hypergraphs which allows us to show that these correlations are limited for a large range of densities. This extends some previous results of Kim, Sudakov and Vu for graphs. From our results we deduce several corollaries on subgraph counts in random d-regular hypergraphs. We also prove a conjecture of Dudek, Frieze, Ruciński and Šileikis on the threshold for the existence of an ℓ-overlapping Hamilton cycle in a random d-regular r-graph. Moreover, we apply our results to prove bounds on the query complexity of testing subgraph-freeness. In the general graphs model, this problem was first studied by Alon, Kaufman, Krivelevich and Ron, who obtained several bounds on the query complexity of testing triangle-freeness.

1 Random regular graphs

While the consideration of random d-regular graphs is very natural and has a long history, this model is much more difficult to analyze than the seemingly similar $\mathcal{G}(n,p)$ and $\mathcal{G}(n,m)$ models due to the dependencies between edges (here $\mathcal{G}(n,p)$ refers to the binomial n-vertex random graph model with edge probability p and $\mathcal{G}(n,m)$ refers to the uniform distribution on all n-vertex graphs with m edges). For small d, the configuration model (due to Bollobás [5]) has led to numerous results on random d-regular graphs. Moreover, the switching method introduced by McKay and Wormald [17] has led to results for a much larger range of d than can be handled by the configuration model. For example, Kim, Sudakov and Vu [15] used such ideas to show that the classical results on distributions of small subgraphs in $\mathcal{G}(n,p)$ carry over to random regular graphs.

In this paper we develop an edge switching technique for random regular r-uniform hypergraphs (also called r-graphs). More precisely, we show that correlations between the existence of edges in a random regular r-graph are small even if we condition on the (non-)existence of some further edges. This allows us to generalise results of Kim, Sudakov and Vu [15] on the appearance of fixed subgraphs in a random regular graph to the hypergraph setting. Moreover, even in the graph case, we can condition on the (non-)existence of a significantly larger edge set than in [15].

A general result of Dudek, Frieze, Ruciński and Šileikis [9] implies that one can transfer many statements from the binomial model to the random regular hypergraph model. This allows them to deduce (from the main result of Dudek and Frieze [7]) the following: if $2 \leq \ell < r$ and $n^{\ell-1} \ll d \ll n^{r-1}$, then a random d-regular r-graph a.a.s. contains an ℓ-overlapping Hamilton cycle, that is, a Hamilton cycle in which consecutive edges overlap in precisely ℓ vertices. They conjectured that the lower bound provides the correct threshold in the following sense:

> *if $2 \leq \ell < r$ and $d \ll n^{\ell-1}$, then a.a.s. a random d-regular r-graph contains no ℓ-overlapping Hamilton cycle.* (1)

Our correlation results allow us to confirm this conjecture. The threshold for a loose Hamilton cycle (i.e. a 1-overlapping Hamilton cycle) in a random d-regular r-graph was recently determined (via

the configuration model) by Altman, Greenhill, Isaev and Ramadurai [3]. This improved earlier bounds by Dudek, Frieze, Ruciński and Šileikis [8]. Altman, Greenhill, Isaev and Ramadurai [3] also investigated the above conjecture and proved that (1) holds under the much stronger condition that $d \ll n$ if $r = 4$ and $d \ll n^{1/2}$ if $r = 3$. The graph case $r = 2$ where d is fixed is a classical result by Robinson and Wormald [20, 21]: if $d \geq 3$ is fixed, then a.a.s. a random d-regular graph has a Hamilton cycle. This was extended to larger d by Cooper, Frieze and Reed [6].

In a similar way, we can transfer several classical counting results for random graphs to the regular setting. We illustrate this for Hamilton cycles, where we extend the density range of a counting result of Krivelevich [16]: for $\log n \ll d \ll n$, a.a.s. the number of Hamilton cycles in a random d-regular n-vertex graph is fairly close to $n!(d/n)^n$. The results by Krivelevich [16] imply the same behaviour for $d \gg e^{(\log n)^{1/2}}$. Similarly, we transfer a general counting result for spanning subgraphs in $\mathcal{G}(n, m)$ due to Riordan [19] to the setting of random regular graphs.

2 Property testing

The running time of any "exact" algorithm that checks whether a given combinatorial object has a given property must be at least linear in the size of the input. Property testing algorithms have the potential to give much quicker answers, although at the cost of not knowing for certain if the desired property is satisfied by the object. A property testing algorithm is usually given oracle access to the combinatorial object, and answers whether the object satisfies the property or is "far" from satisfying it.

To be precise, following e.g. Goldreich, Goldwasser and Ron [11], we define testers as follows. Given a property $\mathcal{P}$, a *tester* for $\mathcal{P}$ is a (possibly randomized) algorithm that is given a distance parameter ε and oracle access to a structure S. If $S \in \mathcal{P}$, then the algorithm must accept with probability at least 2/3. If S is ε-far from $\mathcal{P}$, then the algorithm should reject with probability at least 2/3. If the algorithm is allowed to make an error in both cases, we say it is a *two-sided error tester*; if, on the contrary, the algorithm always gives the correct answer when S has the property, we say it is a *one-sided error tester*.

For graphs (and, more generally, r-graphs) there have been two classical models for testers: one of them is the dense model, and the other is the bounded-degree model. In the dense model, the density of the r-graph is assumed to bounded away from 0, and we say that an r-graph G is ε-far from having property $\mathcal{P}$ if at least εn^r edges have to be modified (added or deleted) to turn G into a graph that satisfies $\mathcal{P}$. Many results have been proved for the dense model. In particular, there exists a characterization of all properties which are testable with constant query complexity (by Alon, Fischer, Newman and Shapira [1] in the graph case and Joos, Kim, Kühn and Osthus [13] in the r-graph case). For the bounded degree graphs model (which assumes that the maximum degree of the input graphs is bounded by a fixed constant), several general results have also been obtained (see for example the results of Benjamini, Schramm and Shapira [4] as well as Newman and Sohler [18]).

Here, we consider the general graphs model and its generalization to r-graphs. In the general graphs model (introduced by Kaufman, Krivelevich and Ron [14]), a graph G with m edges is ε-far from having property $\mathcal{P}$ if at least εm edges have to be modified for the graph to satisfy $\mathcal{P}$. Furthermore, we also assume that the edges are labelled in the sense that for each vertex there is an ordering of its incident edges. It is natural to consider the following two types of queries. Firstly, we allow vertex-pair queries, where any algorithm may take two vertices and ask whether they are

joined by an edge in the graph or not. Secondly, we allow neighbour queries, where any algorithm may take a vertex and ask which vertex is its i-th neighbour.

These notions generalise to hypergraphs in a straightforward way. More precisely, we will consider the following general hypergraphs model, where a hypergraph with m edges is ε-far from having property $\mathcal{P}$ if at least εm edges must be added or deleted to ensure the resulting hypergraph satisfies $\mathcal{P}$. As in the graph case, we will consider two types of queries:

- Vertex-set queries: Any algorithm may take a set of r vertices and ask whether they constitute an edge in the r-graph or not. The answer must be either yes or no.
- Neighbour queries: Any algorithm may take a vertex and ask for its i-th incident edge (according to the labelling of the edges). The answer is either a set of $r-1$ vertices or an error message if the degree of the queried vertex is smaller than i.

In this paper we consider the property $\mathcal{P}$ of being F-free for fixed r-graphs F. In the dense setting, the theory of hypergraph regularity (as developed by Rödl and Skokan [25], Rödl and Schacht [22, 23, 24] as well as Gowers [12]) implies the existence of testers with constant query complexity for this problem.

However, the problem is still wide open for general graphs and hypergraphs. Alon, Kaufman, Krivelevich and Ron [2] studied the problem of testing triangle-freeness. We provide lower and upper bounds for testing F-freeness which apply to large classes of hypergraphs F. In particular, we observe that testing F-freeness cannot be achieved in a constant number of queries whenever F is not a weak forest and the density of the graphs G to be tested is somewhat below the Turán threshold for F. We also provide a different lower bound which improves on the previous for a large range of parameters and r-graphs. Roughly speaking, it provides better bounds than the former if the average degree d of the input r-graph G is not too small. On the other hand, the class of admissible F is more restricted. We also provide three upper bounds on the query complexity.

Kaufman, Krivelevich and Ron [14] also studied the problem of testing bipartiteness in general graphs. It would be interesting to obtain results for the general (hyper)graphs model covering further properties and to improve the lower and upper bounds we present for testing F-freeness.

The formal statements can be found in [10].

References

[1] N. Alon, E. Fischer, I. Newman and A. Shapira, A combinatorial characterization of the testable graph properties: it's all about regularity, *SIAM J. Comput.* **39** (2009), 143–167.

[2] N. Alon, T. Kaufman, M. Krivelevich and D. Ron, Testing triangle-freeness in general graphs, *SIAM J. Discrete Math.* **22** (2008), 786–819.

[3] D. Altman, C. Greenhill, M. Isaev and R. Ramadurai, A threshold result for loose Hamiltonicity in random regular uniform hypergraphs, *arXiv:1611.09423* (2016).

[4] I. Benjamini, O. Schramm and A. Shapira, Every minor-closed property of sparse graphs is testable, *Adv. Math.* **223** (2010), 2200–2218.

[5] B. Bollobás, A probabilistic proof of an asymptotic formula for the number of labelled regular graphs, *European J. Combin.* **1** (1980), 311–316.

[6] C. Cooper, A. Frieze and B. Reed, Random regular graphs of non-constant degree: connectivity and Hamiltonicity, *Combin. Probab. Comput.* **11** (2002), 249–261.

[7] A. Dudek and A. Frieze, Tight Hamilton cycles in random uniform hypergraphs, *Random Structures Algorithms* **42** (2013), 374–385.

[8] A. Dudek, A. Frieze, A. Ruciński and M. Šileikis, Loose Hamilton cycles in regular hypergraphs, *Combin. Probab. Comput.* **24** (2015), 179–194.

[9] A. Dudek, A. Frieze, A. Ruciński and M. Šileikis, Embedding the Erdős-Rényi hypergraph into the random regular hypergraph and Hamiltonicity, *J. Combin. Theory Ser. B* **122** (2017), 719–740.

[10] A. Espuny Díaz, F. Joos, D. Kühn and D. Osthus, Edge correlations in random regular hypergraphs and applications to subgraph testing, *(submitted)* (2018).

[11] O. Goldreich, S. Goldwasser and D. Ron, Property testing and its connection to learning and approximation, *J. ACM* **45** (1998), 653–750.

[12] W. T. Gowers, Hypergraph regularity and the multidimensional Szemerédi theorem, *Ann. of Math. (2)* **166** (2007), 897–946.

[13] F. Joos, J. Kim, D. Kühn and D. Osthus, A characterization of testable hypergraph properties, *arXiv:1707.03303* (2017), extended abstract appeared in IEEE 58th Annual Symposium on Foundations of Computer Science – FOCS 2017, 859–867.

[14] T. Kaufman, M. Krivelevich and D. Ron, Tight bounds for testing bipartiteness in general graphs, *Approximation, randomization, and combinatorial optimization*, vol. 2764 of *Lecture Notes in Comput. Sci.*, 341–353, Springer, Berlin (2003).

[15] J. H. Kim, B. Sudakov and V. Vu, Small subgraphs of random regular graphs, *Discrete Math.* **307** (2007), 1961–1967.

[16] M. Krivelevich, On the number of Hamilton cycles in pseudo-random graphs, *Electron. J. Combin.* **19** (2012), Paper 25, 14.

[17] B. D. McKay and N. C. Wormald, Uniform generation of random regular graphs of moderate degree, *J. Algorithms* **11** (1990), 52–67.

[18] I. Newman and C. Sohler, Every property of hyperfinite graphs is testable, *SIAM J. Comput.* **42** (2013), 1095–1112.

[19] O. Riordan, Spanning subgraphs of random graphs, *Combin. Probab. Comput.* **9** (2000), 125–148.

[20] R. W. Robinson and N. C. Wormald, Almost all cubic graphs are Hamiltonian, *Random Structures Algorithms* **3** (1992), 117–125.

[21] R. W. Robinson and N. C. Wormald, Almost all regular graphs are Hamiltonian, *Random Structures Algorithms* **5** (1994), 363–374.

[22] V. Rödl and M. Schacht, Property testing in hypergraphs and the removal lemma [extended abstract], *STOC'07—Proceedings of the 39th Annual ACM Symposium on Theory of Computing*, 488–495, ACM, New York (2007).

[23] V. Rödl and M. Schacht, Regular partitions of hypergraphs: counting lemmas, *Combin. Probab. Comput.* **16** (2007), 887–901.

[24] V. Rödl and M. Schacht, Regular partitions of hypergraphs: regularity lemmas, *Combin. Probab. Comput.* **16** (2007), 833–885.

[25] V. Rödl and J. Skokan, Regularity lemma for k-uniform hypergraphs, *Random Structures Algorithms* **25** (2004), 1–42.