Vertex partitions of (C_3, C_4, C_6)-free planar graphs

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Abstract

A graph is (k_1, k_2)-colorable if its vertex set can be partitioned into two sets, one inducing a graph with maximum degree at most k_1 and one inducing a graph with maximum degree at most k_2. We show that every (C_3, C_4, C_6)-free planar graph is $(0, 6)$-colorable. We also show that deciding whether a (C_3, C_4, C_6)-free planar graph is $(0, 3)$-colorable is an NP-complete problem.

1 Introduction

A graph is (k_1, k_2)-colorable if its vertex set can be partitioned into a set inducing a graph with maximum degree at most k_1 and a set inducing a graph with maximum degree at most k_2. Choi, Liu, and Oum [1] have established that there exist exactly two minimal sets of forbidden cycle lengths such that every planar graph is $(0, k)$-colorable for some absolute constant k.

- Planar graphs without odd cycles are bipartite, that is, $(0, 0)$-colorable.
- Planar graphs without cycles of length 3, 4, and 6 are $(0, 45)$-colorable.

For every $n \geq 3$, we denote by C_n the cycle on n vertices. To reformulate, if S is any set of cycles, then there exists a k such that every planar graph with no cycle in S is $(0, k)$-colorable if and only if either S contains every odd cycle, or S contains the cycles C_3, C_4, and C_6.

Our aim is to improve the second point. So we are interested in the class $\mathcal{C}$ of (C_3, C_4, C_6)-free planar graphs.

We will sketch the proofs of the following two theorems in the next two sections. Full proofs are available on arXiv [2].

Theorem 1. *Every graph in $\mathcal{C}$ is $(0, 6)$-colorable.*

Theorem 2. *For every $k \geq 1$, either every graph in $\mathcal{C}$ is $(0, k)$-colorable, or deciding whether a graph in $\mathcal{C}$ is $(0, k)$-colorable is an NP-complete problem.*

In addition, we construct a graph in $\mathcal{C}$ that is not $(0, 3)$-colorable in Section 4. This graph and Theorem 2 imply the following.

Corollary 1. *Deciding whether a graph in $\mathcal{C}$ is $(0, 3)$-colorable is an NP-complete problem.*

Since we deal with $(0, k)$-colorings for some $k \geq 2$, we denote by the letter 0 the color of the vertices that induce the independent set and we denote by the letter k the color of the vertices that induce the graph with maximum degree at most k.

2 Sketch of the proof of Theorem 1

The proof will be using the discharging method. For every plane graph G, we denote by $V(G)$ the set of vertices of G, by $E(G)$ the set of edges of G, and by $F(G)$ the set of faces of G.

Let us define the partial order $\preceq$. Let $n_3(G)$ be the number of vertices with degree at least 3 in G. For any two graphs G_1 and G_2, we have $G_1 \prec G_2$ if and only if either $n_3(G_1) < n_3(G_2)$, or $n_3(G_1) = n_3(G_2)$ and $|V(G_1)| < |V(G_2)|$. Note that the partial order $\preceq$ is well-defined and is a partial linear extension of the subgraph poset.

We suppose for contradiction that G is a graph in $\mathcal{C}$ that is not $(0,6)$-colorable and is minimal according to $\preceq$. For all d, let us call a vertex of G of degree d, at most d, and at least d a *d-vertex*, a *d^--vertex*, and a *d^+-vertex* respectively. For all vertex v, a *d-neighbor*, a *d^--neighbor*, and a *d^+-neighbor* of v is a neighbor of v that is a d-vertex, a d^--vertex, and a d^+-vertex respectively. For all d, let us call a face of G of degree d, at most d, and at least d a *d-face*, a *d^--face*, and a *d^+-face* respectively.

We are going to give some weight on the vertices and faces of the graph. Initially, for all d, every vertex or face of degree d has weight $d-4$. Thus every face and every 4^+-vertex has non-negative initial weight. The aim is to transfer some weight from vertices and faces to other vertices and faces so that in the end, every face and every vertex has a non-negative weight, which will lead to a contradiction, since the sum of the weights is negative by Euler's formula.

The general idea is that vertices and faces of high degree will give weight to 2-vertices and 3-vertices. The faces that need the most charge are the 5-faces with many 2-vertices in its boundary.

We will start with some lemmas on the structure of G, that show that aside from 5-faces with many 2-vertices, we can always find the necessary charge locally. Those lemmas can be proven via the following scheme. Assuming the lemma is false, consider a graph H smaller than G with respect to $\preceq$. The graph H admits a $(0,k)$-coloring by minimality of G. We can show that we can extend the coloring to G, a contradiction.

Lemma 1. *G is connected.*

Lemma 2. *G has no 1-vertex.*

Lemma 3. *Every 7^--vertex of G has a 8^+-neighbor.*

Lemma 4. *Every vertex with degree at least 3 and at most 7 has two 8^+-neighbors.*

Lemma 5. *No 3-vertex is adjacent to a 2-vertex.*

Let us now define the concept of *special faces* and *special configurations*, that will correspond to the places that will lack weight the most. A *special face* is a 5-face with three 2-vertices and two non-adjacent 8^+-vertices. See Figure 1, left. A *special configuration* is three 5-faces sharing a common 3-vertex adjacent to three 8^+-vertices, such that all the other vertices of these faces are 2-vertices. See Figure 1, right. We say *special structure* to speak indifferently about a special face or a special configuration.

Let us define a hypergraph $\widehat{G}$ whose vertices are the 8^+-vertices of G and the hyperedges correspond to the sets of 8^+-vertices contained in the same special structure. For every vertex v of $\widehat{G}$, let $\hat{d}(v)$ denote the degree of v in $\widehat{G}$, that is the number of hyperedges containing v.

Lemma 6. *For every vertex v in $\widehat{G}$, $d(v) - \hat{d}(v) \geq 7$.*

Lemma 7. *Every component of $\widehat{G}$ has at least one vertex v such that $d(v) - \hat{d}(v) \geq 8$.*

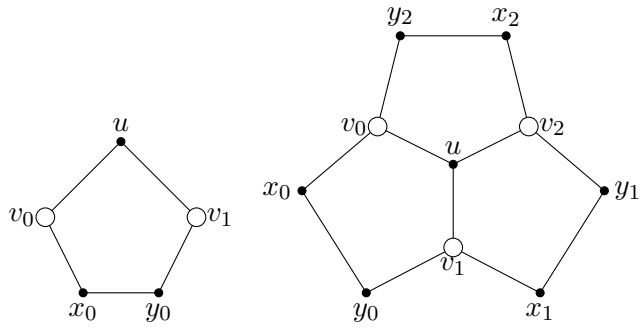


Figure 1: A special face (left) and a special configuration (right).

For each component C of $\widehat{G}$, we choose a vertex v in C such that $d(v) - \hat{d}(v) \geq 8$ as the *root* of C. Now the root v is set to *sponsor* all of the special structures v belongs to. Since C is connected, by a simple search in $\widehat{G}$ we can set every vertex of $C - \{v\}$ to sponsor some of the special structures it belongs to so that every vertex of C aside from v does not sponsor all of the special structures it belongs to, and every special structure has a vertex sponsoring it.

We call a 2-vertex with a 2-neighbor a 2_1-*vertex* and a 2-vertex with no 2-neighbor a 2_0-*vertex*. Some local weight adjustments are omitted here, especially around 5-faces that have low degree vertices in their boundary but are not special structures.

1. Every 8^+-vertex gives weight $\frac{1}{2}$ to each of its 7^--neighbors and to each special structure it sponsors.
2. For each d-vertex v with $3 \leq d \leq 7$ in G, v gives $\frac{1}{2}$ to each of its 2-neighbors.
3. Each 5-face gives $\frac{1}{4}$ to each of its 2_0-vertices and $\frac{5}{8}$ to its 2_1-vertices.
4. Each 7^+-face gives $\frac{3}{4}$ to each of its 2_0-vertices that belong to a 5-face, $\frac{7}{8}$ to each of its 2_1-vertices that belong to a 5-face, $\frac{1}{2}$ to each of its 2_0-vertices that do not belong to a 5-face, and $\frac{3}{4}$ to each of its 2_1-vertices that do not belong to a 5-face.

It can be shown that every vertex and every face has a non-negative final weight.

Let n denote the number of vertices, m the number of edges and f the number of faces of G. By Euler's formula, since G is connected by Lemma 1 and has at least one vertex, $n + f - m = 2$. The initial weight of the graph is $\sum_{v \in V(G)}(d(v) - 4) + \sum_{\alpha \in F(G)}(d(\alpha) - 4) = \sum_{v \in V(G)} d(v) + \sum_{\alpha \in F(G)} d(\alpha) - 4n - 4f = 4m - 4n - 4f = -8 < 0$. Therefore the initial weight of the graph is negative, thus the final weight of the graph is negative, a contradiction. That completes the proof of Theorem 1.

3 Sketch of the proof of Theorem 2

Let $k \geq 3$ be a fixed integer. Suppose that there exists a graph in $\mathcal{C}$ that is not $(0, k)$-colorable. We consider such a graph H_k that is minimal according to $\preceq$. Similarly to Lemmas 1, 2, and 3, the minimum degree of H_k is at least two and every $(k+1)^-$-vertex in H_k is adjacent to a $(k+2)^+$-vertex. A standard discharging argument shows that H_k contains a 2-vertex v. Let u_1 and u_5 be the two neighbors of v.

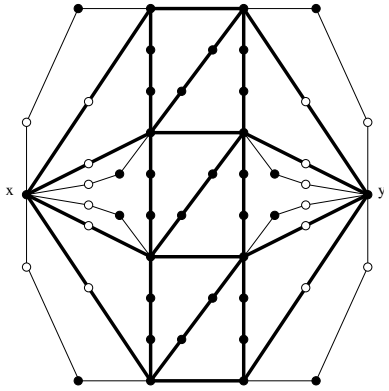


Figure 2: The forcing gadget $F_{x,y}$.

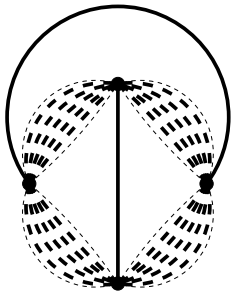


Figure 3: The non-$(0,3)$-colorable graph in $\mathcal{C}$.

By minimality of H_k, $H_k - v$ is $(0,k)$-colorable, every $(0,k)$-coloring of $H_k - v$ is such that u_1 and u_5 get distinct colors, and the vertex in $\{u_1, u_5\}$ that is colored k has exactly k neighbors that are colored k. It can be shown that replacing v in H_k by a path $u_1u_2u_3u_4u_5$ of length 4 leads to a $(0,k)$-colorable graph H'_k in $\mathcal{C}$ and that every $(0,k)$-coloring of H'_k is such that u_3 is colored k and is adjacent to exactly one vertex colored k.

We are ready to prove that deciding whether a graph in $\mathcal{C}$ is $(0,k)$-colorable is NP-complete. The reduction is from the NP-complete problem of deciding whether a planar graph with girth at least 9 is $(0,1)$-colorable [3]. Given an instance G of this problem, we construct a graph $G' \in \mathcal{C}$, as follows. For every vertex v in G, we add $k-1$ copies of H'_k and we add an edge between v and the vertex u_3 of each these copies. The graph G is $(0,1)$-colorable if and only if G' is $(0,k)$-colorable.

4 A graph in $\mathcal{C}$ that is not $(0,3)$-colorable

The graph depicted in Figure 3, where every dashed line represents a copy of the graph $F_{x,y}$ in Figure 2, is a graph in $\mathcal{C}$ that is not $(0,3)$-colorable. The idea is that in every $(0,3)$-coloring of $F_{x,y}$, at least one neighbor of x or y (a white vertex in the picture) is colored 3.

One can see that if we remove the thin parts in Figures 2 and 3, we obtain a planar graph with girth 7 that is not $(0,2)$-colorable. A graph with such properties is already known [4], but this new graph is smaller (184 vertices instead of 1304) and the proof of non-$(0,2)$-colorability is simpler.

References

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[3] L. Esperet, M. Montassier, P. Ochem, and A. Pinlou. A complexity dichotomy for the coloring of sparse graphs. *J. Graph Theory* **73(1)** (2013), 85–102.

[4] M. Montassier and P. Ochem. Near-colorings: non-colorable graphs and NP-completeness. *Electron. J. Comb.* **22(1)** (2015), #P1.57.