Disjoint maximal independent sets in graphs and hypergraphs

Paul Dorbec — **University of Bordeaux, France.**
Fatma Kaci — **University of Biskra, Algeria.**

Abstract

The existence of disjoint maximal independent sets (*disjoint MIS*) in graphs was introduced by C. Berge (unpublished; see [2] and [3]) and independently C. Payan [8]. The study of disjoint MIS was generalized to hypergraphs by Acharya in [1], using the notion of strong independence (i.e. no two vertices belong to the same hyperedge). In this paper we continue these studies. In particular, we prove the existence of disjoint MIS in a connected graph G or in its complement $\overline{G}$ whenever G is of diameter larger than two or G is k-colorable with $k \leq 4$.

1 About disjoint maximal independent sets

In this paper, we are interested in the existence of disjoint maximal independent sets (*disjoint MIS*) in graphs and hypergraphs. Recall that an independent set in a graph is a set of vertices pairwise non adjacent. An independent set S is maximal (inclusion-wise) if there is no superset of S that is also independent. This is equivalent to S being a dominating set. An early study about the existence of disjoint MIS in graphs started with an unpublished conjecture of C. Berge (mentionned in [2, 3]) and independently of C. Payan [8]. They posed the following conjecture:

Conjecture 1. *Every non empty regular simple graph contains two disjoint MIS.*

However, C. Payan found himself a family of counter-examples to this conjecture (see [9], [10]). The smallest graph of the family contains 630 vertices and is of degree 280. Yet, the conjecture was proven to be true for $(n-k)$-regular graphs where $1 \leq k \leq 7$ (see [3]), $k \leq 10$ (see [8]) and for $k < -2 + 2\sqrt{2n}$ (see [4]). Moreover, C. Payan [10] proved also the conjecture in particular for claw-free regular graphs.

Any graph that has an isolate vertex does not admit disjoint MIS, as the isolate is part of any MIS. Another family of graphs that are known not to admit disjoint MIS is formed by coronas of odd cycles.

Based on this observation, O. Schaudt [7] gave a polynomial time algorithm that either computes disjoint MIS or identifies an induced corona of an odd cycle. Note however that having no induced corona of a cycle is not a necessary condition to have disjoint MIS. Actually, Henning *et al.* [5] proved that deciding whether a graph admits disjoint MIS is an NP-complete decision problem.

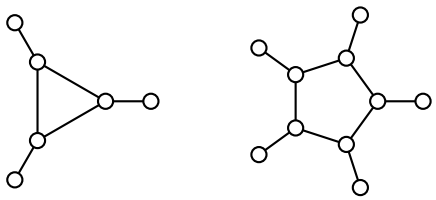


Figure 1: Coronas of odd cycle have no disjoint MIS but have disjoint maximal cliques.

Observe that the above examples of graphs not admitting disjoint MIS admits disjoint maximal complete subgraphs (also called maximal cliques or MCS). Indeed, if a graph as an isolate vertex v, then v itself is a maximal complete subgraph disjoint from any other complete subgraph. In the family of coronas of odd cycles, vertices of degree one are all included in disjoint MCS. Similarly, we do not detail Payan's construction here, but it does contain disjoint MCS (therefore contradicting in the same time a conjecture from [3]). It then seems natural to formulate the following question, that we here explore further :

Question 2. *Let G be a non trivial graph. When is it true that G or $\overline{G}$ admits disjoint MIS? Equivalently, when is it true that either G admits disjoint MIS, or G admits disjoint MCS?*

2 About disjoint MIS or disjoint MCS.

One first direction to consider the question is in regard of the diameter. Note that the only graphs of diameter one are complete graphs, and each vertex is a MIS in the graph. For larger girth, we get the following preliminary result:

Theorem 1. *If $G = (X, E)$ is a connected graph of diameter larger than 2, then G contains disjoint maximal cliques.*

Proof. Consider two vertices u and v at distance at least three in G, and two maximal cliques containing u and v, respectively. If the two cliques share a vertex, it is a common neighbor of u and v, contradicting the assumption on their distance. □

Observe in particular that the above theorem covers the (easy) case when the graph is not connected. Now we only need to consider graphs of diameter 2 in the following. Our next result uses the chromatic number of the graph as a parameter:

Theorem 2. *If a graph G has a chromatic number $\chi(G) \leq 4$, then G has disjoint MIS or disjoint MCS.*

The proof of the above theorem resides on a careful analysis of the partition of the vertices into independent set by an optimal coloring from a particular choice of a greedy coloring algorithm. In particular, if the sets of vertices colored with values larger than one are not maximal independent sets, we identify disjoint maximal cliques.

So we now can infer that the only graphs that may have neither disjoint MIS nor disjoint MCS have diameter two and chromatic number at least 5.

Unfortunately, we figured out that such graphs exists, and we build a family of examples as follows. Start with a complete graph K_p with vertex set $\{x_1, x_2, ..., x_p\}$. For any of the $\binom{p}{2}$ subsets S of $p-2$ vertices of K_p, add a vertex to the graph whose neighborhood is precisely S. The graph G_p thus obtained has $p + \binom{p}{2}$ vertices, diameter 2, and chromatic number p.

Proposition 3. *For $p \geq 4$, G_p has neither disjoint MIS nor disjoint MCS.*

This implies that both above results are best possible in the way they are defined. However, the question of characterizing the graphs with no disjoint MIS nor disjoint MCS remains open.

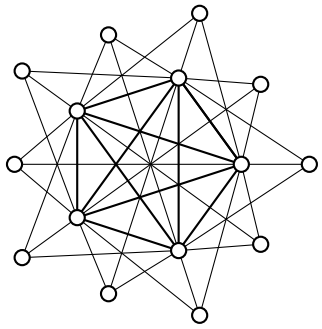


Figure 2: Graph G_5 has neither disjoint MIS nor disjoint maximal cliques.

3 Hypergraphs

The study of disjoint MIS in hypergraphs was started by Acharya [1] where he asks for a characterization of hypergraphs having disjoint MIS [1, Problem 2]. In a hypergraph, a set of vertices is said to be independent if no two vertices belong to the same edge. This notion is sometimes called $\times$*strong-independence* as the generalization of independence to hypergraph is not unique. A cycle in a hypergraph is a sequence $(x_1, E_1, x_2, E_2, \ldots, x_k, E_k)$ of all distinct edges and vertices such that $x_i, x_{i+1} \in E_i$ for all $1 \leq i < k$, with the special case of E_k containing both x_k and x_1. A hypergraph is balanced if every odd cycle uses an edge containing three vertices in the cycle.

Acharya conjectured that every balanced hypergraph admits disjoint MIS. Jose and Tuza disproved that conjecture [6], and suggested that the question might still be relevant for k-connected balanced hypergraphs [6, Problem 4], that is graphs where there are no subset of $k-1$ vertices whose removal would disconnect the graph. Here, we answer to this question by the negative proposing for any k an infinite family of balanced k-connected hypergraphs $\mathcal{H}_{k,t}$ that have no disjoint-MIS.

The hypergraphs $\mathcal{H}_{k,t} = (V, \mathcal{E})$ are constructed as follows: let $V = \{u_{i,j} \mid 1 \leq i \leq k+1, 1 \leq j \leq t\}$ and $\mathcal{E} = \{X_i \mid 1 \leq i \leq k+1, i \neq 2\} \cup \{Y_j, Z_j \mid 1 \leq j \leq t\}$ where $X_i = \{u_{i,1}, \ldots, u_{i,t}\}$, $Y_j = \{u_{3,j}, \ldots, u_{k+1,j}\}$ and $Z_j = \{u_{1,j}, u_{2,j}\}$. Figure 3 illustrates that construction.

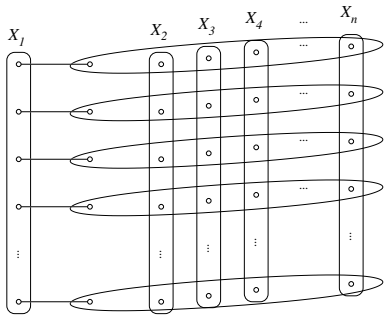


Figure 3: The hypergraph $\mathcal{H}_{k,t}$

Proposition 4. *Let $k \in \mathbb{N}$, $t \geq 2k+1$. The hypergraph $\mathcal{H}_{k,t}$ is balanced, k-connected, and does not have disjoint MIS.*

The proof of this result is based on the following observation. Recall that a 2-packing is a set of vertices which are pairwise at distance at least three and that $N(S)$ is the open-neighborhood of S, i.e. the set of all vertices adjacent to a vertex in S.

Observation 5. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. If there exists a 2-packing S of order ℓ in $\mathcal{H}$ such that $\alpha(H[N(S)]) < \frac{\ell}{2}$, then $\mathcal{H}$ has no disjoint MIS.*

References

[1] B.D. Acharya. Domination in Hypergraph II. New Directions. Advances in discrete mathematics and applications: Mysore, 2008, 1–18, Ramanujan Math. Soc. Lect. Notes Ser., 13, Ramanujan Math. Soc., Mysore, 2010.

[2] J.A. Bondy, U.S.R. Murty. Graph theory with application. American Elseiver Publ Co. New york, 1976.

[3] E.J. Cockayne, S.T. Hedetniemi. Disjoint independent dominating sets in graphs. Discrete Math. 15 (1976), 213–222.

[4] P. Erdös, A.M. Hobbs, C. Payan. Disjoint cliques and disjoint maximal independent sets of vertices in graphs. Discrete Math. 42 (1982), 57–61.

[5] M.A. Henning, C. Löwenstein, D. Rautenbach. Remarks about disjoint domminating sets. Discrete Math. 309 (2009), 6451-6458.

[6] C. Jose, Z. Tuza. Hypergraph domination and strong independence. Appl. Anal. Discrete Math. 3 (2009), no. 2, 347–358.

[7] O. Schaudt. On disjoint Disjoint Maximal Independent Sets in Graphs. Inf. Process. Lett 115(1):23–27 (2015).

[8] C. Payan. Sur une classe de problèmes de couverture. C.R. Acad. Sci. Paris A 278 (1974), 233–235.

[9] C. Payan. A counter-example to the conjecture "Every nonempty regular simple graph contains two disjoint maximal independent sets". Graph Theory Newsletter 6 (1977) 7-8.

[10] C. Payan. Covering by minimal transversals. Discrete Math. 23 (1978) 273-277.