Embedding trees of bounded degree in dense graphs

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Abstract

We conjecture that every n-vertex graph of minimum degree at least $\frac{k}{2}$ and maximum degree at least $2k$ contains all trees with k edges as subgraphs. We prove an approximated version of this conjecture for trees of bounded degree for dense graphs.

Our work also has implications on the Erdős–Sós conjecture and the $\frac{2}{3}$-conjecture. We prove an approximated version of both conjectures for trees of bounded degree and dense host graphs.

1 Introduction

A central problem in graph theory consists of determining which conditions a graph G has to satisfy in order to ensure it contains a given subgraph H. In particular, conditions on the average degree, the median degree or the minimum degree of G that ensure that H embeds in G have been studied extensively. Classical examples include Turán's Theorem, which determines an average degree threshold for the containment of a complete graph, or Dirac's Theorem, which establishes a minimum degree condition for containment of a Hamilton cycle. In this paper we will focus on the embedding of trees in dense graphs.

A greedy embedding argument shows that in a graph G with minimum degree at least k one may embed every tree with k edges. Note that the bound on the minimum degree is tight, as one easily confirms considering the example given by the union of several disjoint copies of K_k (the complete graph on k vertices), which does not contain any tree with k edges.

A classical conjecture of Erdős and Sós states that is possible to replace the minimum degree condition with a similar condition on the average degree. The bound is again tight by the example given above.

Conjecture 1 (Erdős and Sós [3])**.** *Let $k \in \mathbb{N}$ and let G be a graph with average degree greater than $k-1$. Then every tree with k edges is a subgraph of G.*

The Erdős-Sós conjecture is trivially true for stars, and holds for paths by a result of Erdős and Gallai [4]. In the early 1990's Ajtai, Komlós, Simonovits and Szemerédi announced a solution of the Erdős-Sós Conjecture for large graphs.

It is well known that every graph of average degree greater than k has a subgraph of minimum degree greater than $\frac{k}{2}$ and average degree greater than k. So, in order to prove the Erdős–Sós conjecture, it would be enough to prove it with an additional condition on the minimum degree of the host graph.

In this vein, Bollobás [2] conjectured in 1978 that any a graph on n vertices and minimum degree at least $(1+o(1))\frac{n}{2}$ would contain every spanning tree with maximum degree bounded by a constant. This conjecture was verified in 1995, and improved in 2001, by Komlós, Sárközy and Szemerédi [7, 8].

Theorem 1 (Komlós, Sárközy and Szemerédi [8])**.** *For any $\delta \in (0,1)$, there are n_0 and $c \in (0,1)$ such that the following is true for all $n \geq n_0$. Let G be a graph on n vertices and minimum degree at least $(1+\delta)\frac{n}{2}$, then every spanning tree with maximum degree at most $\frac{cn}{\log n}$ is a subgraph of G.*

A natural question is if one can replace n by k in Theorem 1, for some $k < n$, and hope that G would contain every tree with k edges and bounded degree. Clearly, the answer is no, because of the example given before Conjecture 1, or in fact, one could consider the union of disjoint copies of K_ℓ, for any $\lceil \frac{k}{2} \rceil + 1 < \ell \leq k$, provided that ℓ divides the order of G. However, we conjecture that if in addition to the minimum degree condition, we require at least one vertex of large degree in G, then every tree with k edges is contained in G.

Conjecture 2 (Besomi, Pavez-Signé and Stein [1])**.** *Let $k \in \mathbb{N}$ and let G be a graph with minimum degree greater than $\frac{k}{2}$ and maximum degree at least $2k$. Then every tree with k edges is a subgraph of G.*

This is not the first time a combination of a minimum and maximum degree condition has been proposed for replacing the average degree condition in the Erdős–Sós conjecture. In 2016, Havet, Reed, Stein, and Wood put forward the following conjecture [6].

Conjecture 3 (Havet, Reed, Stein and Wood [6])**.** *Let $k \in \mathbb{N}$ and let G be a graph with minimum degree at least $\lfloor \frac{2k}{3} \rfloor$ and maximum degree at least k, then every tree with k edges is a subgraph of G.*

So one might say our Conjecture 2, inspired by Theorem 1, lies halfway between Conjecture 1 and Conjecture 3.

Conjecture 2 is essentially tight due to the following example. Given a sufficiently small $\varepsilon > 0$, let G_ε consist of two copies of the complete bipartite graph with parts of size $(1-\varepsilon)k$ and $(1-\varepsilon)\frac{k}{2}$, and one vertex that is adjacent to every vertex in the parts of size $(1-\varepsilon)k$. It is easy to see that G_ε does not contain the tree T_k consisting of $\sqrt{k}$ stars of size $\sqrt{k}$ whose centers are adjacent to the central vertex of T_k, provided that k is sufficiently large depending on ε.

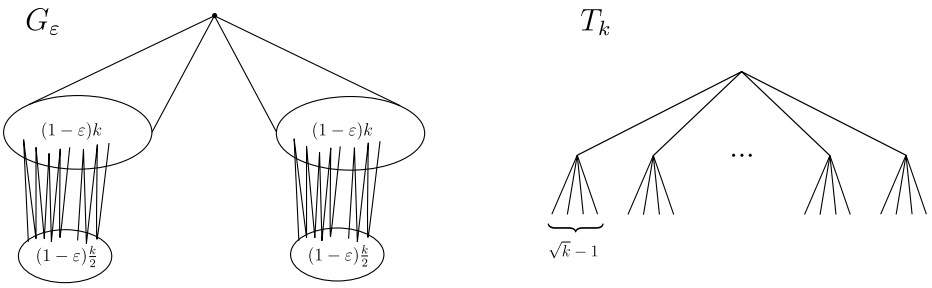


Figure 1: Extremal example for Conjecture 2

We prove an approximated version of Conjecture 2 for trees of bounded degree and dense host graphs.

Theorem 2. [1] *For each $\delta > 0$ there exists k_0 such that for every $k \geq k_0$ and every $n \in \mathbb{N}$ with $n > k > \delta n$ the following holds. Every graph G on n vertices with minimum degree at least $(1+\delta)\frac{k}{2}$ and maximum degree at least $(1+\delta)2k$ contains every tree with k edges whose maximum degree is bounded by k^c, where $c = \frac{\delta}{24+3\delta}$.*

Using the same method as in the proof of Theorem 2, we are also able to prove approximated versions both of Conjecture 1 and of Conjecture 3, for bounded degree trees and dense host graphs.

Theorem 3. [1] *Let $\delta > 0$, there is k_0 such that for every $k \geq k_0$ and every $n \in \mathbb{N}$ with $n > k > \delta n$ the following holds. Every graph G on n vertices and average degree greater than $(1+\delta)k - 1$ contains every tree with k edges whose maximum degree is bounded by k^c, where $c = \frac{\delta}{24+3\delta}$.*

Theorem 4. [1] *Let $\delta > 0$, there is k_0 such that for every $k \geq k_0$ and every $n \in \mathbb{N}$ with $n > k > \delta n$ the following holds. Every graph G on n vertices with maximum degree at least $(1+\delta)k$ and minimum degree at least $(\frac{2}{3}+\delta)k$ contains every tree with k edges whose maximum degree is bounded by k^c, where $c = \frac{\delta}{24+3\delta}$.*

2 Sketch of the proof of Theorem 2

Let G be a graph satisfying the assumptions of Theorem 2. Our proof follows an approach based on the regularity lemma of Szemerédi, which is commonly used for embedding problems in dense graphs.

Let $0 < \beta \ll \varepsilon \ll \eta \ll \delta$. We apply the regularity lemma on G, with parameter ε, in order to obtain an ε-regular partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_\ell$. The (ε, η)-reduced graph of G is defined as the graph with vertex set $\{V_i : i \in [\ell]\}$, called *clusters*, in which V_iV_j is an edge if and only if (V_i, V_j) is an ε-regular pair with density greater than η. The reduced graph inherits several useful properties of G, for instance it has approximately the same edge density of G. However, it might fail to be connected even if G has high connectivity.

Let R be the (ε, η)-reduced graph of G, the minimum degree of G implies that the minimum degree of R is at least $(1+\frac{\delta}{2})\frac{\ell}{n} \cdot \frac{k}{2}$. If R has at least one large connected component we can use the following key proposition.

Proposition 1. *Set $\alpha = \frac{k}{n}$ and let C be a connected component of R that has diameter at most d.*

(i) If C is bipartite and has one part of size at least $(1+\frac{\delta}{2})\alpha\ell$, then every k-edge tree with maximum degree at most $k^{\frac{1}{d+1}}$ is contained in G.

(ii) If C is non-bipartite and has size at least $(1+\frac{\delta}{2})\alpha\ell$, then every k-edge tree with maximum degree at most $k^{\frac{1}{4d+3}}$ is contained in G.

We remark that if Proposition 1 can be applied, that is, if R has at least one large component C, then we can take $c = \frac{\delta}{24+3\delta}$ in Theorem 2 because of a result of Erdős, Pach, Pollack and Tuza [5], which implies that the diameter of C is at most $6\delta^{-1} - 1$.

So, now assume that Proposition 1 cannot be applied. This means that every non-bipartite component of R has size less than $(1+\frac{\delta}{2})\alpha\ell$ and every bipartite component has size at most $(1+\frac{\delta}{2})2\alpha\ell$. Let x be a vertex of maximum degree in G and let $z \in T$ such that every component of $T - z$ has size at most $\frac{k}{2}$. We may map z into x and then develop some ad hoc methods to show one can embed $T - z$ into the components of R seen by x. We refer to [1] for all details.

So, it only remains to show Proposition 1.

Sketch of a proof of Proposition 1. Let T be a tree with k edges and degree bounded by k^c, where $c = \frac{1}{d+1}$ if C is bipartite and $c = \frac{1}{4d+3}$ otherwise. We decompose T into a constant number of subtrees of size less than βk, called *pieces*, and we order these subtrees in a way that every initial segment under this order forms a subtree in T.

Assume that $C = A \cup B$ is bipartite and that $|A| \geq (1 + \frac{\delta}{2})\alpha\ell$. We embed T piece by piece following the prescribed ordering. This will be done in a way that the heavier part of T will be embedded into A. Assume without loss of generality that the root of T is in the heavier part, then we start the embedding by mapping the root into A. Suppose we are embedding a piece F, we can always choose an edge $XY \in E(C)$ which has enough free space in order to embed F into $X \cup Y$. This is possible since A is large enough and since the minimum degree of any cluster is at least $(1 + \frac{\delta}{2})\frac{\alpha\ell}{2}$. Choose a minimal path P in C that connects XY with the last edge used during the embedding. Observe that P has length at most d and every pair of consecutive clusters in P is ε-regular with density greater than η. Using regularity we can embed the first $d-1$ levels of F across P, which uses only a tiny fraction of vertices in $V(P)$. The remaining vertices can embedded in $X \cup Y$ because XY was chosen to have enough free space for F. This process is repeated until we finish the embedding.

If C is non-bipartite, we claim that there exists a cluster-matching[1] M of size at least $(1+\frac{\delta}{2})\alpha\ell$. We decompose T into pieces as before, so that the embedding will be done piece by piece in an ordered way. Denote by ϕ the partial embedding that is defined iteratively. At every step of the embedding we may choose an edge $XY \in M$, with many unused vertices, so that we can embed a new piece F into $X \cup Y$. To do so, we need to guarantee that

$$||\phi^{-1}(A)| - |\phi^{-1}(B)|| \leq \frac{\varepsilon n}{\ell} \text{ for all } AB \in M, \tag{1}$$

at every step of the embedding, so that we fill every edge of M in a balanced way. It turns out that we may ensure (1) if we can choose which colour class of F will be embedded in A and B, respectively. This can be done by using a minimal odd cycle in order to build an odd/even path from XY to the previous edge used in the embedding. The embedding of F is done across the path and into $X \cup Y$, as in the bipartite case. For all further details of the proof, we refer to [1]. □

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[1] Actually one can find a matching M together with a family of disjoint triangles. Then, by subdividing every cluster into two parts of the same size one gets the desired matching.