Color Patterns by Dynamic $H-$walks

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Abstract

Let H be a digraph possibly with loops, let D be a multidigraph, and let $c : A(D) \to V(H)$ be a coloring of the arcs of D with the vertices of H. A sequence $W = (x_0, x_1, \ldots, x_n)$ in D, is a dynamic H-walk if for each $i \in \{0, \ldots, n-2\}$ there exist f_i and f_{i+1} such that f_i is an arc from x_i to x_{i+1}, f_{i+1} is an arc from x_{i+1} to x_{i+2}, and $(c(f_i), c(f_{i+1}))$ is an arc of H. For $u, v \in V(D)$, we say that u reaches v by a dynamic H-walk if there exists a dynamic H-walk from u to v in D. A subset $K \subseteq V(D)$ is a dynamic H-kernel of D if every vertex in $V(D) - K$ reaches some vertex in K by a dynamic H-walk, and no vertex in K can be reached by another vertex in K by a dynamic H-walk in D. Let $\mathscr{D}_3$ be the family of digraphs H such that for every H-colored digraph D, D has a dynamic H-kernel. In this talk, we provide a characterization of $\mathscr{D}_3$.

Throughout this work we will consider two types of digraphs; patterns will be digraphs without parallel arcs, but possibly with loops and symmetric arcs. A pattern will be usually denoted by H, and its set of vertices will be used to color the arcs of a digraph D, which might contain parallel arcs, loops and symmetric arcs (so D is an H-arc-colored multidigraph). We will always consider finite digraphs.

Given a pattern H and an H-arc-colored multidigraph D (with arc coloring $c : A(D) \to V(H)$), Arpin and Linek considered kernels by H-walks in [1]. A walk $W = (x_0, x_1, \ldots, x_n)$ of D is an H-walk if $c(W) = (c((x_0, x_1)), c((x_1, x_2)), \ldots, c((x_{n_1}, x_n)))$ is a walk in H. Defining in an obvious way reachability by H-walks, we also have the definitions of independence and absorbance by H-walks, or H-independence and H-absorbance for short, and naturally, kernels by H-walks or H-kernels. Based on this definitions, Arpin and Linek defined the class $\mathscr{B}_3$ as the set of digraphs H such that for every H-arc-colored multidigraph D, D has an H-kernel. In 2016, Galeana-Sánchez and Strausz [3] characterized the digraphs in $\mathscr{B}_3$ as the digraphs that are bicomplete or can be contracted to $2K_2$.

Following the work of Arpin and Linek, the concept of H-walk can be generalized in the following way. We will allow "lane changes", this is, instead of considering a single H-walk, we will allow concatenation of two H-walks as long as the last arc of the first one and the first arc of the second one have the same head and tail (recall that D is a multidigraph). So, a new concept of reachability is obtained in this way, instead of following a single H-walk to reach one vertex from another, we might "change lanes" to keep advancing. Formally, let H be a pattern and D a multidigraph. An H-arc-coloring of D is a function $\zeta : A(D) \to V(H)$ such that for each pair of different vertices $x, y \in V(D)$ and each $c \in V(H)$ there is at most one arc, say f, with head x and tail y such that $\zeta(f) = c$. Hence, for any $x, y \in V(D)$, we may assume that there are at most $|V(H)|$ arcs in D from x to y. Also, notice that ζ does not have to be surjective. For $x, y \in V(D)$, we define the set $A_D[x, y]$ as $A_D[x, y] = \{f \in A(D) \colon\ f \text{ has head } x \text{ and tail } y\}$. A *dynamic H-walk* in an H-arc-coloured multidigraph D is thus a sequence $W = (x_0, \ldots, x_n)$ such that, for each $i \in \{0, \ldots, n-2\}$ there exist $f_i \in A[x_i, x_{i+1}]$ and $f_{i+1} \in A[x_{i+1}, x_{i+2}]$ with $(\zeta(f_i), \zeta(f_{i+1})) \in A(H)$. If W does not repeat vertices, then it is a *dynamic H-path*. For vertices $x, y \in A(D)$, the existence of an xy-dynamic H-walk in D will be denoted by $u \xrightarrow{dynamic} v$; if there is no such walk, we will write

$u \xrightarrow{dynamic} v$. It follows from the definition of dynamic H-walk that the color can be changed in every step, this makes dynamic H-walks able to model more real life situations than H-walks.

Based on the dynamic H-walks definition, a subset $S \subseteq V(D)$ is a *dynamic H-absorbent set* of D if for each $x \in V(D) - S$ there exist $s \in S$ such that $x \xrightarrow{dynamic} s$, and it is a *dynamic H-independent* set of D whenever for any $x, y \in S$ $x \xrightarrow{dynamic} y$. A *dynamic H-kernel* of D is a dynamic H-absorbent and dynamic H-independent subset of $V(D)$.

In the first part of this work, we will give examples that illustrate the difference between $H-$walks and dynamic $H-$walks, as well as the difference between $H-$kernel and dynamic $H-$kernel. Hence, we conclude that the concept of dynamic $H-$kernel is indeed different from that of $H-$kernel and, in particular, their associated decision problems are also different. Nonetheless, the similarities between these concepts are enough to let us easily reduce the problem of finding a dynamic $H-$kernel in an $H-$arc-colored digraph to the problem of finding an $\hat{H}$-kernel in an $\hat{H}$-arc-colored digraph for a particular digraph $\hat{H}$.

Theorem 1. *Let H be a pattern and D an H-arc-colored multidigraph. Then there exist a pattern $\hat{H}$, and an $\hat{H}$-arc-colored multidigraph $\hat{D}$ such that D has a dynamic H-kernel if and only if $\hat{D}$ has an $\hat{H}$-kernel.*

As in [1], we define $\mathscr{D}_3$ as the class of digraphs H such that for every H-arc-colored multidigraph D, D has a dynamic H-kernel. The following lemma gives us some of the properties of $\mathscr{D}_3$.

Lemma 1. *Let H be a pattern. The following properties hold:*

1. *If $H \in \mathscr{D}_3$, then every vertex of H has a loop.*
2. *If $H \in \mathscr{D}_3$ and H' is an induced subdigraph of H, then $H' \in \mathscr{D}_3$.*

Let H and H' be digraphs such that every vertex of H and every vertex of H' has a loop, a *full homomorphism* from H to H' is a function $\varphi : V(H) \to V(H')$ such that for any pair of vertices $u, v \in V(H)$, we have $(u, v) \in A(H)$ if and only if $(\varphi(u), \varphi(v)) \in A(H')$. Let $V_1, \ldots, V_k$ be the inverse images of vertices in H', thus, since H and H' are looped, the inverse image of any vertex of H' is either empty or a looped complete digraph. Also, if we let $V_i = \varphi^{-1}(x)$ and $V_j = \varphi^{-1}(y)$, it follows from the definition that there is an arc from V_i to V_j if and only if $(x, y) \in A(H')$, moreover, $V_i \times V_j \subseteq A(H)$ if and only if $(x, y) \in A(H')$. Following Arpin and Linek [1], we say that H' is a *contraction* of H if there exists a surjective full homomorphism from H to H'.

Lemma 2. *Let H' be a contraction of H, then $H \in \mathscr{D}_3$ if and only if $H' \in \mathscr{D}_3$.*

Clearly, K_1 (with a loop) belongs to $\mathscr{D}_3$, because every digraph has a kernel by paths [2]. In [6], Sands, Sauer and Woodrow proved that $2K_1$ (with a loop on each vertex) belongs to $\mathscr{B}_3$, but we will prove that it does not belong to $\mathscr{D}_3$. Which brings us to the next theorem.

Theorem 2. *The digraph $2K_1$ is not an element of $\mathscr{D}_3$. Therefore, if $H \in \mathscr{D}_3$, then H is a semicomplete digraph.*

Using these properties we can make a complete analysis of patterns in $\mathscr{D}_3$, where we obtain a complete classification of the digraphs on three vertices in $\mathscr{D}_3$, which leads to a structural characterization of those digraphs in $\mathscr{D}_3$.

Let M be an $n \times n$ matrix with entries in $\{0, 1, *\}$. We say that a digraph D is M-partitionable if $V(D)$ admits a partition (as it is usual in graph theory, some parts may be empty) $(V_1, \ldots, V_n)$ such that

- All the arcs from V_i to V_j are present in D if $M_{ij} = 1$.
- None of the arcs from V_i to V_j are present in D if $M_{ij} = 0$.
- There are no restrictions for the arcs from V_i to V_j if $M_{ij} = *$.

Notice that, in particular, when $M_{ii} = 0$, then V_i is an independent set, and when $M_{ii} = 1$, then V_i is a complete digraph. The different problems that arise from the concept of M-partition are explored in depth in the fantastic survey [4]. In [3], Galeana-Sánchez and Strausz proved that a pattern H belongs to $\mathscr{B}_3$ if and only if H is either $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$-partitionable, or $\begin{pmatrix} 1 & 1 \\ * & 1 \end{pmatrix}$-partitionable. In [5], Hell and Hernández-Cruz proved the following result.

Lemma 3. *A digraph H is $\begin{pmatrix} 1 & 1 \\ * & 1 \end{pmatrix}$-partitionable if and only if it does not contain $2\overleftrightarrow{K}_1$, C_3, T_3, or $\mathcal{F}_3$ as an induced subdigraph.*

Hence, the following theorem follows from Lemma 3 together with the results and the discussion in the previous part of this work.

Theorem 3. *Let H be a pattern, then $H \in \mathscr{D}_3$ if and only if it is $\begin{pmatrix} 1 & 1 \\ * & 1 \end{pmatrix}$-partitionable.*

In conclusion, we have introduced the concept of dynamic H-walk, and used it to define dynamic H-independence, dynamic H-absorbance and dynamic H-kernels, as an analogue to H-kernels.

As one would expect, dynamic H-kernels and H-kernels behave differently; we exhibited, patterns H and H-arc-colored multidigraphs having either an H-kernel and no dynamic H-kernel, or a dynamic H-kernel but no H-kernel. Nonetheless, we noticed that under certain transformations of both H and D, we can make these to problems "equivalent". These observations led to a complete classification of the dynamic analogue of the *panchromatic patterns* deffined in [3], this is, the patterns in $\mathscr{D}_3$. It is worth noticing that the set $\mathscr{D}_3$ turned out to be different to the set $\mathscr{B}_3$ of panchromatic patterns. It is hard to have the correct intuition with dynamic H-kernels with respect to H-kernels; one would think that, since it is "easier" to find a dynamic H-walk than an H-walk, and thus, it would be "easier" to absorb a vertex and find a dynamic H-kernel than an H-kernel, hence, the set $\mathscr{D}_3$ would contain the set $\mathscr{B}_3$. Nonetheless, the set $\mathscr{D}_3$ is properly contained in $\mathscr{B}_3$.

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