Recognizing minimally tough graphs

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Abstract

Let t be a positive real number. A graph is called t-tough, if the removal of any cutset S leaves at most $|S|/t$ components. The toughness of a graph is the largest t for which the graph is t-tough. A graph is minimally t-tough, if the toughness of the graph is t and the deletion of any edge from the graph decreases the toughness. The complexity class DP is the set of all languages that can be expressed as the intersection of a language in NP and a language in coNP. We proved that recognizing minimally t-tough graphs is DP-complete for any positive integer t and for any positive rational number $t \leq 1$. We also investigate the minimum degree and the recognizability of minimally t-tough graphs in the class of chordal graphs, split graphs, claw-free graphs and $2K_2$-free graphs.

1 Introduction

All graphs considered in this paper are finite, simple and undirected. Let $\omega(G)$ denote the number of components and $\alpha(G)$ denote the independence number.

Definition 1. *Let t be a positive real number. A graph is called* t-tough*, if the removal of any cutset S leaves at most $|S|/t$ components. The* toughness *of a graph is the largest t for which the graph is t-tough.*

Definition 2. *A graph is* minimally t-tough*, if the toughness of the graph is t and the deletion of any edge from the graph decreases the toughness.*

It follows directly from the definition of toughness that every t-tough graph is $2t$-connected, therefore, the minimum degree of any t-tough graph is at least $\lceil 2t \rceil$.

The motivation for our research is the following conjecture.

Conjecture 1 ([3])**.** *Every minimally* 1*-tough graph has a vertex of degree* 2*.*

A natural generalization of the conjecture is that every minimally t-tough graph has a vertex of degree $\lceil 2t \rceil$. The conjecture is still open, but in [4] we proved that every minimally 1-tough graph has a vertex of degree at most $n/3 + 1$.

In [4] we showed that in the class of claw-free graphs the conjecture is true in a very strong sense: the only minimally 1-tough, claw-free graphs are cycles. On the other hand, the class of minimally t-tough graphs is large for any positive rational number t: any graph can be embedded as an induced subgraph into a minimally t-tough graph [4].

It is natural to ask, how "large" is the set of minimally t-tough graphs for different t values and for various graph classes. Now we investigate this question from a complexity viewpoint.

Bauer et al. proved that for any positive rational number t, recognizing t-tough graphs is coNP-complete [1]. The focus of our investigation is on recognizing minimally tough graphs.

The complexity class DP was introduced by Papadimitriou and Yannakakis [7].

Definition 3. *A language L is in the class* DP *if there exist two languages $L_1 \in NP$ and $L_2 \in coNP$ such that $L = L_1 \cap L_2$.*

A language is called DP-hard if all problems in DP can be reduced to it in polynomial time. A language is DP-complete if it is in DP and it is DP-hard.

In our proofs we use a critical-type DP-complete problem, α-Critical [8]. In this problem a graph G and a positive integer k are given and we want to decide if $\alpha(G) < k$, and $\alpha(G - e) \geq k$ for any edge $e \in E(G)$.

2 Recognizing minimally tough graphs

Our main result is that we prove that recognizing minimally t-tough graphs is DP-complete for some values of t.

Theorem 1. *Recognizing minimally t-tough graphs is DP-complete for any positive integer t and for any positive rational number $t \leq 1$.*

To show that this problem is DP-hard for every positive integer t and for every rational number $1/2 < t \leq 1$, we reduce α-Critical to it. For every positive rational number $t \leq 1/2$, we reduce a slightly modified problem of Min-1-Tough to it.

For the remaining values the problem is open.

Conjecture 2. *Recognizing minimally t-tough graphs is DP-complete for any positive rational number t.*

We also study chordal, split, claw-free and $2K_2$-free graphs. The interesting property of these graph classes is that they are not closed for edge-deletion. We handle this behavior by showing that for each edge e of these graphs there exists a vertex set $S(e)$ guaranteed by the following claim with some nice properties.

Claim 1. *Let t be a positive rational number and G a minimally t-tough graph. For every edge e of G,*

1. *the edge e is a bridge in G, or*
2. *there exists a vertex set $S = S(e) \subseteq V(G)$ with*

$$\omega(G - S) \leq \frac{|S|}{t} \quad \text{and} \quad \omega\big((G - e) - S\big) > \frac{|S|}{t},$$

and the edge e is a bridge in $G - S$.

In the first case, we define $S = S(e) = \emptyset$.

3 Chordal graphs

It is easy to prove that every chordal graph has a simplicial vertex, i.e. a vertex whose neighborhood forms a clique. The main idea of the proofs of the following two theorems is that in a minimally t-tough, chordal graph with $t \leq 1$ no vertex sets guaranteed by Claim 1 can contain a simplicial vertex.

Theorem 2. *For any rational number $1/2 < t \leq 1$, there exist no minimally t-tough, chordal graphs.*

Theorem 3. *Let $t \leq 1/2$ be a positive rational number. If G is a minimally t-tough, chordal graph, then every simplicial vertex of G has degree 1.*

Thus, the generalized Kriesell conjecture is true for minimally t-tough, chordal graphs with $t \leq 1$.

4 Split graphs

The toughness of split graphs can be computed in polynomial time, see [5] and [9].

Let G be a split graph partitioned into a clique C and an independent set I. If an edge goes between C and I, then after its removal the graph is still a split graph, so we can compute in polynomial time whether its toughness decreased. The following claim says that for any positive rational number t, the vertex sets showing that the removal of any edge in the clique decreases the toughness can be determined.

Claim 2. *Let G be a minimally t-tough, split graph, partitioned into a clique C and an independent set I. Let $e = uv$ be an edge between two vertices of C and $S = S(e) \subseteq V(G)$ a vertex set guaranteed by Claim 1. Then*

$$S = \big(C \setminus \{u, v\}\big) \cup \{w \in I \mid uw, vw \in E(G)\}.$$

Theorem 4. *For any positive rational number t, minimally t-tough, split graphs can be recognized in polynomial time. More precisely, if G is a minimally t-tough, split graph partitioned into a clique C and an independent set I, then there exists an integer $b > 1$ for which $t = 1/b$, and $|C| \leq 3$, moreover,*

1. *either G is a tree with at most two internal vertices and with $\Delta(G) = b$,*
2. *or $|C| = 3$, every vertex in I has degree 1 and every vertex in C has degree $b + 1$.*

Thus, the generalized Kriesell conjecture is true for minimally tough, split graphs.

5 Claw-free graphs

By the following theorem, the toughness of claw-free graphs can be computed in polynomial time.

Theorem 5 ([6])**.** *If G is a noncomplete claw-free graph, then $2\tau(G) = \kappa(G)$.*

Now we give a characterization of minimally t-tough graphs with $t \leq 1$. For $t = 1$ we can show that there exists a vertex set guaranteed by Claim 1 of size at most 2. For $t = 1/2$ we can show that there exist such vertex sets of size at most 1.

Theorem 6 ([4])**.** *The class of minimally 1-tough, claw-free graphs are the cycles of length at least 4.*

Theorem 7. *The class of minimally $1/2$-tough, claw-free graphs are exactly those connected graphs that can be built up in the following way.*

1. *Take a tree T with maximum degree 3 where the set of vertices of degree 1 and 3 together form an independent set.*

2. *Now delete every vertex of degree 3, but connect its 3 neighbors with a triangle.*

Corollary 1. *Every minimally $1/2$-tough, claw-free graph has a vertex of degree 1.*

Thus, the generalized Kriesell conjecture is true for minimally t-tough, claw-free graphs with $t \leq 1/2$.

6 $2K_2$-free graphs

The toughness of $2K_2$-free graphs can be computed in polynomial time. Let G be a minimally t-tough, $2K_2$-free graph. Then we can prove that for every edge there exists a vertex set guaranteed by Claim 1, which is contained by the open neighborhood of the endpoints of the edge (i.e. the set of vertices adjacent to any of the endpoints excluding the endpoints themselves).

Theorem 8. *For any positive rational number t, the class of minimally t-tough, $2K_2$-free graphs can be recognized in polynomial time.*

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