Decomposability of graphs into subgraphs fulfilling the 1–2–3 Conjecture

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Abstract

The well-known 1–2–3 Conjecture asserts that the edges of every graph without isolated edges can be weighted with 1, 2 and 3 so that adjacent vertices receive distinct weighted degrees. This is open in general. We prove that every d-regular graph, $d \geq 2$, can be decomposed into at most 2 subgraphs (without isolated edges) fulfilling the 1–2–3 Conjecture if $d \notin \{10, 11, 12, 13, 15, 17\}$, and into at most 3 such subgraphs in the remaining cases. Additionally, we prove that in general every graph without isolated edges can be decomposed into at most 24 subgraphs fulfilling the 1–2–3 Conjecture, improving the previously best upper bound of 40. Both results are partly based on applications of the Lovász Local Lemma.

1 Introduction

A graph of order at least 2 cannot be *irregular*, *i.e.* its vertices cannot have pairwise distinct degrees. Potential alternative definitions of "irregular graphs" (referring to more local features of a graph) were investigated by Chartrand, Erdős and Oellermann in [9], while the related so-called *irregularity strength* of a graph was introduced in [10], and investigated further in numerous papers, see *e.g.* [3, 11, 14, 17, 19]. The following notion was in turn proposed in [5]: A graph (or multigraph) is called *locally irregular* if its **adjacent** vertices have distinct degrees. We say a graph $G = (V, E)$ can be *decomposed into k locally irregular subgraphs* if E can be partitioned into k sets: $E = E_1 \cup E_2 \cup \ldots \cup E_k$ so that $G_i := (V, E_i)$ is locally irregular for $i = 1, 2, \ldots, k$. In [5] it was conjectured that except for some family of *exceptional graphs* (each of which has maximum degree at most 3, see [5] for details), every connected graph can be decomposed into 3 locally irregular subgraphs. This was then confirmed in [20] for graphs with sufficiently large minimum degree.

Theorem 1 ([20])**.** *Every graph G with minimum degree $\delta(G) \geq 10^{10}$ can be decomposed into* 3 *locally irregular subgraphs.*

It was also proved by Bensmail, Merker and Thomassen [8] that in general every connected graph which is not exceptional can be decomposed into (at most) 328 locally irregular subgraphs, what was then pushed down to 220 such subgraphs in [18]. See also [5, 6, 8, 18] for a number of partial and related results.

An another direction of research towards inducing local irregularities in graphs was earlier introduced in [16]. There Karoński, Łuczak and Thomason considered the least k so that a locally irregular multigraph can be obtained from a given graph $G = (V, E)$ by replacing each edge with at most k parallel edges. This problem was in fact originally formulated in terms of weightings, where by a *k-edge-weighting* of G we mean any mapping $\omega : E \to \{1, 2, \ldots, k\}$. For such w we may define the so-called *weighted degree of* or simply the *sum at* a given vertex v as: $s_\omega(v) := \sum_{e \in E_v} \omega(e)$, where E_v denotes the set of edges incident with v in G. In this setting, the authors of [16] were interested in the least k such that a k-edge-weighting ω of G exists so that $s_\omega(u) \neq s_\omega(v)$ for every edge $uv \in E$ – we say that u and v are *sum-distinguished* then (note we must assume that G contains no isolated edges to that end). They posed a very intriguing question, commonly known as the *1–2–3 Conjecture* in the literature nowadays, and confirmed it *e.g.* for 3-colourable graphs.

Conjecture 2 (1–2–3 Conjecture, [16]). *For every graph $G = (V, E)$ without isolated edges there exists a weighting $\omega : E \to \{1, 2, 3\}$ sum-distinguishing all neighbours in G.*

Theorem 3 ([16]). *Every graph G with no isolated edges and $\chi(G) \leq 3$ fulfills the 1–2–3 Conjecture.*

In general Conjecture 2 is however still widely open. The best general result thus far was delivered by Kalkowski, Karoński and Pfender [15], who proved that it is sufficient to use weights $1, 2, 3, 4, 5$, see also [1, 2] for other results.

We develop research initiated in [7], related to and combaining the both concepts discussed above. In the following, by writing that a graph *fufills the 1–2–3 Conjecture*, we shall mean that there actually exists its neighbour sum-distinguishing 3-edge-weighting (assuming this holds in particular for an edgeless graph). We prove below that for almost every $d \geq 2$, a d-regular graph G can be decomposed into 2 subgraphs *fulfilling the 1–2–3 Conjecture*, while in the remaining cases it can be decomposed into 3 such subgraphs, *cf.* Corollary 6 and Theorem 7 below. We also show that in general every graph without isolated edges can be decomposed into (at most) 24 subgraphs consistent with the 1–2–3 Conjecture, while thus far it was known that 40 such subgraphs were always sufficient, see [7] (also for other related results). Our approach is in both cases partly based on various applications of the probabilistic method, and in particular symmetric and general versions of the Lovász Local Lemma, see *e.g.* [4], and the Chernoff Bound, see *e.g.* [12].

2 Graphs with Bounded Chromatic Number

We first analyse graphs with relatively small degrees by investigating, more generally, graphs with upper-bounded chromatic number. For this aim we develop research from [7] to prove that:

Lemma 4. *For each positive integer k, every graph $G = (V, E)$ without isolated edges and with $\chi(G) \leq 3^k$ can be decomposed into k (some possibly empty) subgraphs $G_1, G_2, \ldots, G_k$ such that $\chi(G_i) \leq 3$ and G_i contains no isolated edges for $i = 1, 2, \ldots, k$.*

By Theorem 3 and Lemma 4 we thus almost immediately obtain the following corollaries.

Corollary 5. *Every graph G without isolated edges can be decomposed into $\lceil \log_3 \chi(G) \rceil$ graphs fulfilling the 1–2–3 Conjecture.*

Corollary 6. *Let G be a d-regular graph. If $2 \leq d \leq 9$, then G can be decomposed into 2 subgraphs fulfilling the 1–2–3 Conjecture, while if $10 \leq d \leq 27$, then G can be decomposed into 3 subgraphs fulfilling the 1–2–3 Conjecture.*

3 Main Result for Regular Graphs

Our main result concerning regular graphs is the following.

Theorem 7. *Every d-regular graph G with $d \geq 14$, $d \neq 15, 17$, can be decomposed into two graphs fulfilling the 1–2–3 Conjecture.*

In order to prove Theorem 7, we first apply the symmetric version of the Local Lemma to prove in Lemma 8 the existence of a specific partition $V = V_0 \cup V_1$ of the vertex set of a given graph $G = (V, E)$. This is then used within the proof of Theorem 7 itself to construct a decomposition of

G into two subgraphs G_0, G_1 (where G_i contains all edges induced by V_i in G and part of the edges joining V_i with V_{1-i} for $i = 0, 1$) consistent with the assumption of Lemma 9 (and thus fulfilling the 1–2–3 Conjecture). To prove Lemma 9 in turn we apply a certain refinement of Kalkowki's algorithm from [13], exploiting for this aim the concept of maximal independent sets.

Lemma 8. *The vertices of every d-regular graph G with $d \geq 14$, $d \neq 15, 17$, can be partitioned into sets V_0 and V_1 such that if $d \equiv r \bmod 2$ for some $r \in \{0, 1\}$, then:*

(i) $\forall v \in V_0 : d_{V_0}(v) \geq 2 + r$; *(ii)* $\forall v \in V_0 : d_{V_1}(v) \geq 2$;

(iii) $\forall v \in V_1 : d_{V_1}(v) \geq 2 + r$; *(iv)* $\forall v \in V_1 : d_{V_0}(v) \geq 2$.

Lemma 9. *If a graph $G = (V, E)$ contains a maximal independent set I such that there exists a constant $\alpha \geq 1$ so that for $R := V \setminus I$,*

(1°) $d(v) \leq \alpha$ *for every* $v \in I$ *and*

(2°) $d(v) \geq \alpha + \frac{d_R(v)+1}{2}$ *for every* $v \in R$,

then G fulfills the 1–2–3 Conjecture.

4 General Upper Bound for All Graphs

For general graphs, *i.e.* not necessarily regular, we prove the following.

Theorem 10. *Every graph G without isolated edges can be decomposed into* 24 *subgraphs fulfilling the 1–2–3 Conjecture.*

The proof of this relies strongly on the following lemma (partly inspired by [8]), whose thesis could be optimized via application of the general version of the Local Lemma (used to prove an additional auxiliary lemma).

Lemma 11. *Every graph $G = (V, E)$ without isolated edges can be decomposed into two graphs H and F such that: H is either empty or has minimum degree $\delta(H) \geq 10^{10}$, and F contains no isolated edges and has degeneracy less than $10^{10} + 10^8$.*

Theorem 10 then follows, as by Lemma 11 any graph without isolated edges can be decomposed into H and F such that by Theorem 1, H can be further decomposed into 3 locally irregular subgraphs (which fulfill the 1–2–3 Conjecture), and by Lemma 5, F can be decomposed into 21 subgraphs fulfilling the 1–2–3 Conjecture.

5 Concluding Remarks

Note that by Corollary 6 and Theorem 7 we know that any d-regular graph without isolated edges can be decomposed into 2 subgraphs fulfilling the 1–2–3 Conjecture if only $d \notin \{10, 11, 12, 13, 15, 17\}$. The remaining cases apparently need a separate special treatment, but either way, by Corollary 6 every d-regular graph, $d \geq 2$, can be decomposed into (at most) 3 subgraphs complying with the 1–2–3 Conjecture. It is however believed that something stronger should hold. Namely, it was conjectured that every graph G without isolated edges and isolated triangles can be decomposed into 2 subgraphs admitting neighbour sum-distinguishing 2-edge-weightings, see [7] for details and further observations concerning this new concept, and many other related problems and results.

References

[1] L. Addario-Berry, K. Dalal, C. McDiarmid, B.A. Reed, A. Thomason, Vertex-Colouring Edge-Weightings, Combinatorica 27(1) (2007) 1–12.

[2] L. Addario-Berry, K. Dalal, B.A. Reed, Degree Constrained Subgraphs, Discrete Appl. Math. 156(7) (2008) 1168–1174.

[3] M. Aigner, E. Triesch, Irregular assignments of trees and forests, SIAM J. Discrete Math. 3(4) (1990) 439–449.

[4] N. Alon, J.H. Spencer, The Probabilistic Method, 2nd edition, Wiley, New York, 2000.

[5] O. Baudon, J. Bensmail, J. Przybyło, M. Woźniak, On decomposing regular graphs into locally irregular subgraphs, European J. Combin. 49 (2015) 90–104.

[6] O. Baudon, J. Bensmail, É. Sopena, On the complexity of determining the irregular chromatic index of a graph, J. Discrete Algorithms 30 (2015) 113–127.

[7] O. Baudon, J. Bensmail, T. Davot, H. Hocquard, J. Przybyło, M. Senhaji, É. Sopena, M. Woźniak, A general decomposition theory for the 1–2–3 Conjecture and locally irregular decompositions, submitted (https://hal.archives-ouvertes.fr/hal-01690222).

[8] J. Bensmail, M. Merker, C. Thomassen, Decomposing graphs into a constant number of locally irregular subgraphs, European J. Combin. 60 (2017) 124–134.

[9] G. Chartrand, P. Erdős, O.R. Oellermann, How to Define an Irregular Graph, College Math. J. 19(1) (1988) 36–42.

[10] G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz, F. Saba, Irregular networks, Congr. Numer. 64 (1988) 197–210.

[11] A. Frieze, R.J. Gould, M. Karoński, F. Pfender, On Graph Irregularity Strength, J. Graph Theory 41(2) (2002) 120–137.

[12] S. Janson, T. Łuczak, A. Ruciński, Random Graphs, Wiley, New York, 2000.

[13] M. Kalkowski, A note on 1,2-Conjecture, in Ph.D. Thesis, Poznań, 2009.

[14] M. Kalkowski, M. Karoński, F. Pfender, A new upper bound for the irregularity strength of graphs, SIAM J. Discrete Math. 25 (2011) 1319–1321.

[15] M. Kalkowski, M. Karoński, F. Pfender, Vertex-coloring edge-weightings: Towards the 1-2-3 conjecture, J. Combin. Theory Ser. B 100 (2010) 347–349.

[16] M. Karoński, T. Łuczak, A. Thomason, Edge weights and vertex colours, J. Combin. Theory Ser. B 91 (2004) 151–157.

[17] J. Lehel, Facts and quests on degree irregular assignments, Graph Theory, Combinatorics and Applications, Willey, New York, 1991, 765–782.

[18] B. Lužar, J. Przybyło, R. Soták, New bounds for locally irregular chromatic index of bipartite and subcubic graphs, submitted (https://arxiv.org/abs/1611.02341).

[19] T. Nierhoff, A tight bound on the irregularity strength of graphs, SIAM J. Discrete Math. 13(3) (2000) 313–323.

[20] J. Przybyło, On decomposing graphs of large minimum degree into locally irregular subgraphs, Electron. J. Combin. 23(2) (2016) ♯P2.31.