Sequential Metric Dimension[1]

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Abstract

Seager introduced the following game in 2013. An invisible and immobile target is hidden at a vertex t of a graph G. Every step, one vertex of G can be probed, informing us of its distance to t. The goal is to locate t, whatever vertex it be, in the minimum number of steps.

We address the generalization of this game where $k \geq 1$ vertices can be probed at every step. Precisely, given a graph G and two integers $k, \ell \geq 1$, the Localization Problem asks if there exists a strategy to locate a target hidden in G in at most ℓ steps by probing at most k vertices per step. We show that this problem is NP-complete when k (resp., ℓ) is a fixed parameter.

In the class of trees, we show that Localization is NP-complete when k and ℓ are part of the input. Despite this, we design a polynomial-time $(+1)$-approximation algorithm in trees. It follows that Localization is polynomial-time solvable in trees if k is fixed.

1 Introduction

Localization (or *Identification*) *problems* consist of making the vertices of a graph $G = (V, E)$ distinguishable using a smallest subset $R \subseteq V$ of vertices. Many variants have been considered, where the main difference is the distinction condition that R must provide. One notable example, connected to our investigations in this work, is the one of *resolving sets* [HM76, Sla75], where R is resolving whenever any two vertices of G can be distinguished according to their distances to the vertices in R. The minimum size $MD(G)$ of a resolving set of G is called the *metric dimension* of G [HM76, Sla75]. In general, computing $MD(G)$ is NP-complete (even when restricted to planar graphs and diameter-2 graphs), and W[2]-hard when parameterized by the solution's size (see [FMN$^+$17b] and the bibliography there). On the positive side, the problem is FPT for bounded-treelength graphs [BFGR17], while some general bounds on the metric dimension have been exhibited for some graph classes [FMN$^+$17a].

In this paper, we address a *sequential variant* of the problem of determining the metric dimension of a graph. Let us consider a graph $G = (V, E)$ where an unknown vertex $t \in V$ hosts a hidden (invisible) and immobile target. *Probing* one vertex $v \in V$ results in the knowledge of the distance between t and v in G, denoted by $d_G(v, t)$. Probing a set $R \subseteq V$ of vertices results in a *distance vector* $(d_G(v, t))_{v \in R}$. Rephrased differently, R is resolving if it does not yield two vertices of G having the same distance vector, while $MD(G)$ is the minimum number of vertices that must be probed simultaneously to determine t. As simple examples, note that for any path P we have $MD(P) = 1$ (just probe one end), while for any star S_n with n leaves we have $MD(S_n) = n - 1$ (it is necessary and sufficient to probe all leaves but one).

If less than $MD(G)$ vertices can be probed at once, it is natural to allow more than one probing step. Obviously, if at most $1 \leq k < MD(G)$ vertices can be probed at once, then it is always

[1]The full version of this paper can be found online at `http://www-sop.inria.fr/members/Fionn.Mc-Inerney/localization.pdf`.

possible to locate an immobile target in at most $\lceil MD(G)/k \rceil$ steps by sequentially probing k different vertices of a smallest resolving set at each step. However, there are graphs for which the target can be located much faster. In [Sea13], Seager studied this aspect under a game formalism, where the immobile target must be located as fast as possible by probing one vertex each step. She notably gave bounds and exact values on the minimum number of probing steps needed in particular subclasses of trees (e.g., subdivisions of caterpillars). In this paper, we study the generalization of this game where $k \geq 1$ vertices can be probed at every step.

Precisely, let $k \geq 1$ be an integer and $G = (V, E)$ be a graph hosting an invisible and immobile target hidden at $t \in V$. By a *k-strategy*, we mean a sequence of probing steps resulting in the determination of t, where, at each step, at most k vertices are probed. Note that, in general, in a k-strategy the choice of the vertices to be probed at some step is highly dependent of the results obtained through the previous probing steps. We denote by $\lambda_k(G)$ the minimum integer h such that there exists a k-strategy resulting in the determination of t in at most h steps, whatever t be. For a given graph G and two integers $k, \ell \geq 1$, the Localization Problem asks whether $\lambda_k(G) \leq \ell$. We also consider the dual parameter $\kappa_\ell(G)$ defined as the minimum integer h such that there exists an h-strategy to locate t in at most ℓ steps. Note that, for every graph G, the parameter $\kappa_1(G)$ is exactly the metric dimension $MD(G)$ of G, and $\lambda_k(G) \leq \ell$ if and only if $\kappa_\ell(G) \leq k$.

1.1 Related work

Moving target. Sequential games related to resolving sets have first been introduced and mainly studied in the case of a mobile target. That is, at every step, some vertices may be probed and, if the target has not been located yet, it may move to one of its neighbouring vertices (sometimes, though, it is required that the target cannot "backtrack", *i.e.*, move to a vertex that has been probed during the previous step). In this setting, locating the target is not always feasible. For instance, if the moving target is allowed to backtrack, then it is not possible to locate it in a triangle by probing only one vertex each step. In [Sea12] and [BDE$^+$17], this game was studied in the context where the graph is a tree and the target cannot backtrack. In [Sea14], the case of trees when the target is allowed to backtrack was considered. Let $\zeta(G)$ be the minimum integer k such that there exists a k-strategy for locating a moving target in G. In [BGG$^+$17b], it was shown that deciding whether $\zeta(G) \leq k$ is NP-hard and that $\zeta(G)$ is not bounded in the class of graphs with treewidth 2. There, it was also proved that $\zeta(G) \leq 3$ for any outerplanar graph G.

Relative distance and centroidal dimension. Foucaud *et al.* defined a variant of resolving sets, called *centroidal bases*, where the vertices of a graph $G = (V, E)$ must be distinguished by their *relative distances* to the probed vertices [FKS14]. In this setting, given an integer $k \geq 2$, probing a set $B = \{v_1, \cdots, v_k\}$ of vertices results in a *relative-distance vector* $(\delta_{i,j}(t))_{1 \leq i < j \leq k}$ where, for every $1 \leq i < j \leq k$, $\delta_{i,j}(t) = 0$ if $d_G(t, v_i) = d_G(t, v_j)$, $\delta_{i,j}(t) = 1$ if $d_G(t, v_i) > d_G(t, v_j)$ and $\delta_{i,j}(t) = -1$ otherwise. Intuitively speaking, this vector, for any vertex t, gives an ordering over the v_i's indicating which ones are the closest to t. The set B is a centroidal basis if the relative-distance vectors are different for any two vertices of G. The *centroidal dimension* of G, denoted by $CD(G)$, is the minimum size of a centroidal basis of G [FKS14]. Note that $CD(G) \geq 2$ in general (unless G has order 1), and that $CD(G) \leq |V|$ (as, clearly, V is a centroidal basis of G). In [FKS14], it was proved that determining the centroidal dimension of a graph is NP-complete in general, and almost tight bounds on the centroidal dimension of paths have been computed.

Just as for the metric dimension, sequential variants of centroidal bases can naturally be defined.

The variant with a mobile target was introduced in [BGG+17a]. Here, we also initiate the study of the variant where the target is immobile. Let $k \geq 2$ be an integer and G be a graph. We denote by $\lambda_k^{rel}(G)$ the minimum integer h such that there exists a k-strategy for locating (through the relative-distance vectors) a hidden immobile target in G within at most h steps, wherever it be. Given G, k, ℓ, the Relative-Localization Problem asks whether $\lambda_k^{rel}(G) \leq \ell$. The dual parameter $\kappa_\ell^{rel}(G)$ is defined as the minimum integer h such that there exists an h-strategy for locating (through the relative-distance vectors) the target in G in at most ℓ steps. Note that, for every graph G, the parameter $\kappa_1^{rel}(G)$ is exactly the centroidal dimension $CD(G)$ of G, and $\lambda_k^{rel}(G) \leq \ell$ if and only if $\kappa_\ell^{rel}(G) \leq k$.

2 Our results

Throughout this work, every graph considered is connected, undirected, and simple. Our first results are about the computational complexity of Localization and Relative-Localization. In particular, we show that they are NP-complete when k or ℓ is fixed; precisely:

Theorem 1. *Let $k \geq 1$ (resp. $k' \geq 2$) be a fixed integer. Given a graph G and an integer $\ell \geq 1$ as inputs, deciding whether $\lambda_k(G) \leq \ell$ (resp. $\lambda_{k'}^{rel}(G) \leq \ell$) is NP-complete.*
Let $\ell \geq 1$ be a fixed integer. Given a graph G and an integer $k \geq 1$ (resp. $k' \geq 2$) as inputs, deciding whether $\kappa_\ell(G) \leq k$ (resp. $\kappa_\ell^{rel}(G) \leq k'$) is NP-complete.

The main result of this paper is the comprehension of Localization in trees. As a first step, we show that, in general, the problem remains hard for trees.

Theorem 2. Localization *is NP-complete when restricted to trees.*

However, we show that, surprisingly, Localization becomes polynomial-time solvable after the first step. This follows from the following observation. Let us consider a tree T where a target is hidden and assume that, as a first step, a single node $r \in V(T)$ is probed. After this single probe, the distance $d \in \mathbb{N}$ between the target and r is known. Therefore, from the second step, the instance becomes equivalent to a tree T' (a subtree of T) rooted in r, whose leaves are all the same distance d from r, and where the target is known to occupy a leaf of T'. Our main result is a proof that, in such instances, the target can be located in polynomial time independent of k and ℓ.

Theorem 3. *There is an algorithm with running time $O(n \log n)$ that, given an integer $k \geq 1$ and an n-node tree T, computes a k-strategy that locates a target in T in at most $\lambda_k(T) + 1$ steps.*

In brief words, this algorithm runs as follows. It first probes any one node, so that, as explained earlier, for the next step the instance is reduced to a tree T' with the convenient properties described above. Let $v_1, \cdots, v_d$ be the children of r, and, for any v_i, let T_{v_i} denote the subtree of T' rooted at v_i. One key argument used by the algorithm is that, for any integer $1 \leq i \leq d$, it suffices to probe one node in any T_{v_i} to determine whether the target is in T_{v_i} or $T' - T_{v_i}$.

From this, locating the target is then just determining in which subtree it is located and recursively repeating this process for that subtree. But, if enough nodes are probed in T_{v_i}, then it is advantageous if the target is in T_{v_i} as it is as if the first turn of an optimal strategy in T_{v_i} were played and thus, there only remain $\lambda_k(T_{v_i}) - 1$ steps to locate the target in T_{v_i}. We denote by $\pi(T_{v_i})$, the minimum number of nodes needed to be probed in T_{v_i} (the first time a node is probed in T_{v_i}), so that $\lambda_k(T_{v_i}) - 1$ steps remain to locate the target in T_{v_i} if it is in T_{v_i}.

Precisely, the algorithm functions by dynamic programming. For each subtree T_{v_i}, it calculates $\pi(T_{v_i})$, as well as an optimal strategy (using $\lambda_k(T_{v_i})$ steps) to locate the target on a leaf of T_{v_i}. The subtrees are ordered from largest to smallest in terms of the lexicographical ordering of $(\lambda_k(T_{v_i}), \pi(T_{v_i}))$. Our algorithm tests these subtrees in this order by either probing 1 or $\pi(T_{v_i})$ nodes in each subtree T_{v_i} (if the target is located in a subtree before testing each subtree T_{v_i}, then subtrees of that subtree are now tested). We prove that the algorithm determines the optimal number of nodes to be probed, that probing them in this order is an optimal strategy, and thus, that $(\lambda_k(T_r), \pi(T_r))$ can be "easily" calculated.

As explained earlier, the hardness of LOCALIZATION in trees arises from the first step. However, it can be solved greedily by considering all possible first probings, and then applying our algorithm, which is optimal starting from the second step, for each such resulting instance. This yields:

Theorem 4. *There is an algorithm with running time $O(n^{k+2} \log n)$ that, given an integer $k \geq 1$ and an n-node tree T, computes a k-strategy that locates a target in T in at most $\lambda_k(T)$ steps.*

3 Further work

Our results in trees leave the open question of whether LOCALIZATION is FPT (in k) in the class of trees T. Also, we do not know the complexity of determining whether $\kappa_\ell(T) \leq k$ for a tree T. It could be interesting to study these problems in other graph classes, like interval or planar graphs.

RELATIVE-LOCALIZATION seems to be much more intricate even for simple topologies. A first step towards a better understanding of it would be to fully understand the centroidal dimension of paths (*i.e.*, to determine $\kappa_1^{rel}(P)$ for every path P), which has been initiated in [FKS14]. A more challenging direction would then be to consider the case of all trees.

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