Discrete Morse theory for the collapsibility of supremum sections

Balthazar Bauer — **INRIA, DIENS, PSL research, CNRS, Paris, France**
Lucas Isenmann — **LIRMM, Université de Montpellier, CNRS, Montpellier, France**

Abstract

The Dushnik-Miller dimension of a poset $\leq$ is the minimal number d of linear extensions $\leq_1, \dots, \leq_d$ of $\leq$ such that $\leq$ is the intersection of $\leq_1, \dots, \leq_d$. Supremum sections are simplicial complexes introduced by Scarf [13] and are linked to the Dushnik-Miller as follows: the inclusion poset of a simplicial complex is of Dushnik-Miller dimension at most d if and only if it is included in a supremum section coming from a representation of dimension d. Collapsibility is a topological property of simplicial complexes which has been introduced by Whitehead [18] and which resembles shellability. While Ossona de Mendez [12] proved that a particular type of supremum sections are shellable, we show in this article that supremum sections are in general collapsible thanks to the discrete Morse theory developped by Forman [8].

1 Introduction

The order dimension (also known as the Dushnik-Miller dimension) of a poset $\leq$ has been introduced by Dushnik and Miller [4]. It is defined as the minimum number d of linear extensions $\leq_1, \dots, \leq_d$ of $\leq$ such that $\leq$ is the intersection of these extensions *i.e.* $\forall x, y \in V, x \leq y \iff (\forall i \in [\![1, d]\!], x \leq_i y)$. See [16] for a comprehensive study of this topic. This notion is important because, for example, of a theorem of Schnyder [14] which states that a graph is planar if and only if the Dushnik-Miller dimension of the inclusion poset of the associated simplicial complex is at most 3. Representations were introduced by Scarf [13]. A d-representation on a set V is a set of d linear orders on V. Given R a representation on a set V, we can define a simplicial complex $\Sigma(R)$ associated to this representation that we call its supremum section. Scarf proved that every supremum section of a representation satisfying some additional properties, so called "standard", is the inclusion poset of a d-polytope with one face removed. Ossona de Mendez [12] proved that every abstract simplicial complex of Dushnik-Miller dimension at most d is contained in a complex which is shellable and has a straight line embedding in $\mathbb{R}^{d-1}$. Supremum sections also appeared in commutative algebra: Bayer *et al.* [2] studied monomial ideals which are linked to supremum sections by what they call Scarf complexes. They are used by Felsner *et al.* [7] in order to study orthogonal surfaces. They also appear in the study of Gonçalves *et al.* [9] of a variant of Delaunay graphs and in the study of empty rectangles graphs by Felsner [6]. Furthermore, they also appear in spanning-tree-decompositions and in the box representations problem as shown by Evans *et al.* [5].

The goal of our article is to generalize the result of Ossona de Mendez about the shellability of standard supremum sections to every supremum sections. As there exists supremum sections which are not shellable, for instance the simplicial complex characterized by its facets $\{a, b, c\}$ and $\{c, d, e\}$, we will replace shellability by collapsibility which is a similar notion. A collapse is a topological operation on simplicial complexes, and more generally on CW-complexes, introduced by Whitehead [18] in order to define a simple homotopy equivalence which is a refinement of the homotopy equivalence. A complex is said to be collapsible if it collapses to a point. See [10] for a comprehensive study of this topic. The discrete Morse theory introduced by Forman [8] is based on this notion and has numerous applications in applied mathematics and computer science. Homotopy equivalence is a topological notion of topological spaces introduced to classify topological

spaces. Roughly speaking, two spaces are said to be homotopy equivalent if there exists a continuous deformation from one to the other. A topological space is said to be contractible if it is homotopy equivalent to a point. Collapsible spaces form an important subclass of contractible spaces. While contractibility is algorithmically undecidable by a result of Novikov [17], the subclass of collapsible spaces is algorithmically recognizable. More precisely Tancer [15] showed that it is NP-complete to decide whether a simplicial complex is collapsible. Furthermore, every 1-dimensional contractible complex is collapsible but the house with two rooms [1] and the dunce hat [19] show that there are complexes which are contractible but not collapsible. Finally, the conjecture of Zeeman [19], which implies the Poincarré conjecture, states that for every finite contractible 2-dimensional CW-complex K, the space $K \times [0,1]$ is collapsible.

2 Notations

In the following, V is a finite set. An *(abstract) simplicial complex* Δ is a subset of $\mathcal{P}(V)$ closed by inclusion (*i.e.* $\forall X \in \Delta, \forall Y \subseteq X, Y \in \Delta$). We call *faces* the elements of Δ and *facets* the maximal faces of Δ according to the inclusion order.

Definition 1 (Ossona de Mendez [12])**.** *Given a linear order $\leq$ on a set V, an element $x \in V$, and a set $F \subseteq V$, we say that x dominates F in $\leq$, and we denote it $F \leq x$, if $f \leq x$ for every $f \in F$. A* d-representation *R on a set V is a set of d linear orders $\leq_1, \ldots, \leq_d$ on V. Given a d-representation R, an element $x \in V$, and a set $F \subseteq V$, we say that x dominates F in R if x dominates F in some order $\leq_i \in R$. We define $\Sigma(R)$ as the set of subsets F of V such that every $v \in V$ dominates F in R. The set $\Sigma(R)$ is called the* supremum section *of R.*

It is easy to show that if R is a d-representation on a set V, then $\Sigma(R)$ is a simplicial complex. An example is the following 3-representation on $\{a,b,c,d,e\}$: $a <_1 b <_1 e <_1 d <_1 c$, $c <_2 b <_2 a <_2 d <_2 e$, and $e <_3 d <_3 c <_3 b <_3 a$. The corresponding complex $\Sigma(R)$, depicted on the left of Figure 1, is characterized by its facets $\{a,b\}, \{b,c,d\}$, and $\{b,d,e\}$. For example $\{a,b,c\}$ is not in $\Sigma(R)$ as b does not dominate $\{a,b,c\}$ in any order.

Definition 2. *Let Δ be a simplicial complex. We say that a face F of Δ is a* free face *of Δ if it is non-empty, non-maximal and contained in only one facet of Δ.*

Let Δ and Γ be two simplicial complexes. We say that Δ collapses *to Γ if there exists k simplicial complexes $\Delta_1, \ldots, \Delta_k$ and a free face F_i of Δ_i for every $i \in [\![1, k-1]\!]$ such that $\Delta_1 = \Delta$, $\Delta_{i+1} = \Delta_i \setminus \{F \in \Delta_i : F_i \subseteq F\}$ for every $i \in [\![1, k-1]\!]$ and $\Delta_k = \Gamma$. We say that Δ is* collapsible *if it collapses to a point.*

The Hasse diagram of a poset is the transitive reduction of the digraph of the poset. Let R be a representation on a set V, we denote $H(R)$ the Hasse diagram of the inclusion poset of $\Sigma(R)$.

Definition 3. *Let $(\leq, V)$ be a poset and let M be a matching of the Hasse diagram of $\leq$. For an arc a of the Hasse diagram of $\leq$, we denote $d(a)$ and $u(a)$ the elements of V such that $a = (u(a), d(a))$ and $d(a) < u(a)$. A matching M of the Hasse diagram of $\leq$ is said to be* acyclic *if, when reversing the orientation of the arcs of M, the Hasse diagram remains acyclic.*

It is known that if $\leq$ is the poset of inclusion of a simplicial complex and M is a matching of the Hasse diagram of $\leq$ then M is acyclic if and only if there is no sequence of arcs $m_1, \ldots, m_n$ of M such that $(u(m_{i+1}), d(m_i))$ is in the Hasse diagram for all $i \in [\![1, n-1]\!]$ as well as $(u(m_1), d(m_n))$.

Theorem 4 (Chari [3])**.** *Let Δ be a simplicial complex. If the Hasse diagram of the inclusion poset of Δ admits a complete (*i.e. *perfect) acyclic matching, then Δ is collapsible.*

3 Our contribution

Theorem 5. *Let R be a representation on a set V. Then $\Sigma(R)$ is collapsible.*

Because of Theorem 4, it is enough to show that if R is a representation on a set V, then $H(R)$ admits a complete acyclic matching.

3.1 Sketch of proof

Here we only give a sketch[1] of the proof of Theorem 5.

The proof relies on an induction on the dimension of the representation R. The base case is true because if $R = (\leq_1)$ is a 1-representation on a set V, then $\Sigma(R) = \{\emptyset, \{m_1\}\}$ where $m_1 = \min_{\leq_1} V$ and $(\{m_1\}, \emptyset)$ is a complete acyclic matching of $H(R)$.

Let $d \geq 2$ such that the result is true for any $(d-1)$-representation on a set V. Let R be a d-representation on a set V. We denote $R' = (\leq_1, \ldots, \leq_{d-1})$ the $(d-1)$-representation on V obtained from R by deleting the order $\leq_d$. We can show that the simplicial complex $\Sigma(R')$ is a subcomplex of $\Sigma(R)$. See Figure 1 to see what $\Sigma(R')$ is for the example representation of Definition 1.

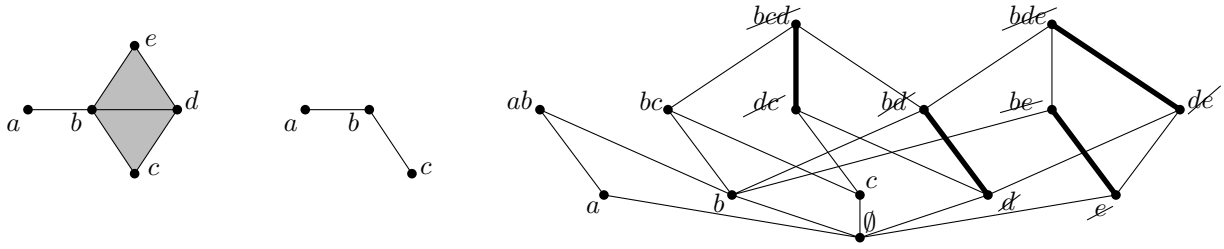


Figure 1: R denotes the representation from the example. From left to right: $\Sigma(R)$ where a grey section corresponds to a face with 3 elements, $\Sigma(R')$ where R' is the representation obtained from R by deleting the order $\leq_3$, the Hasse diagram of $\Sigma(R)$ where the crossed-out faces are the faces from $\Sigma(R) \setminus \Sigma(R')$ and where the thick edges correspond to a complete acyclic matching of the Hasse diagram of the inclusion poset of $\Sigma(R) \setminus \Sigma(R')$.

We show that the following function ψ is well defined:

$$\begin{array}{rccl} \psi & : & \Sigma(R) \setminus \Sigma(R') & \to & V \\ & & F & \mapsto & \min_{<_d} \{x \in V : x <_i (\max_{\leq_i} F) \ \ \forall i \in [\![1, d-1]\!]\} \end{array}$$

We can therefore define the sets $A = \{F \in \Sigma(R) \setminus \Sigma(R') : \psi(F) \notin F\}$ and $B = \{F \in \Sigma(R) \setminus \Sigma(R') : \psi(F) \in F\}$. The goal is to find a complete acyclic matching between A and B. See Figure 1 to see an example of a complete acyclic matching of the Hasse diagram of the inclusion poset of $\Sigma(R) \setminus \Sigma(R')$.

We define a matching M_1 on the Hasse diagram of the inclusion poset of $\Sigma(R) \setminus \Sigma(R')$ by $M_1 = \{(F \cup \{\psi(F)\}, F) : F \in A\}$. We can show that M_1 is well-defined and complete.

[1] A full version of this manuscript including the complete proof can be found at `https://arxiv.org/abs/1803.09577`.

Suppose that M_1 is not acyclic: there exists a sequence $m_1, \dots, m_n$ of arcs of M_1. Thus there exists a sequence $F_1, \dots, F_n$ of A such that $m_i = (F_i \cup \{\psi(F_i)\}, F_i)$ for every $i \in [\![1, n]\!]$ such that $(F_{i+1} \cup \{\psi(F_{i+1})\}, F_i)$ is in the Hasse diagram for every $i \in [\![1, n-1]\!]$ as well as $(F_1 \cup \{\psi(F_1)\}, F_n)$. For every $i \in [\![1, n-1]\!]$, $F_i \subseteq F_{i+1} \cup \psi(F_{i+1})$ and $|F_i| + 1 = |F_{i+1} \cup \psi(F_{i+1})|$, and we deduce that $\psi(F_{i+1}) \in F_i$. We can therefore show that $\psi(F_{i+1}) <_d \psi(F_i)$ for every $i \in [\![1, n-1]\!]$ and thus that $\psi(F_n) <_d \psi(F_1)$. As $(F_1 \cup \{\psi(F_1)\}, F_n)$ is in the Hasse diagram, we show in the same way that $\psi(F_1) <_d \psi(F_n)$ which contradicts the fact that $\psi(F_n) <_d \psi(F_1)$. We conclude that M_1 is a perfect acyclic matching of the Hasse diagram of the inclusion poset of $\Sigma(R) \setminus \Sigma(R')$.

By induction hypothesis, $H(R')$ admits a complete acyclic matching M_2. Finally, we can show that $M_1 \cup M_2$ is a complete acyclic matching of $H(R)$ and we conclude by induction.

References

[1] R.H. Bing, Some aspects of the topology of 3-manifolds related to the Poincaré conjecture, Lectures on Modern Mathematics , II , Wiley, pp. 93–128, 1964.

[2] D. Bayer, I. Peeva, B. Sturmfels, Monomial resolutions, Mathematical Research Letters 5, 31 – 46, 1998.

[3] M. K. Chari, On discrete Morse functions and combinatorial decompositions. Discrete Mathematics, 217(1-3), 101-113, 2000.

[4] B. Dushnik and E.W. Miller, Partially ordered sets, *American Journal of Mathematics*, 63(3): 600-610, 1941.

[5] W. Evans, S. Felsner, G. Kobourov, and T. Ueckerdt, Graphs admitting d-realizers: spanning-tree-decompositions and box-representations *Proc. of EuroCG 14*, 2014.

[6] S. Felsner, Empty Rectangles and Graph Dimension, *ArXiv preprint* arXiv:math/0601.767, 2006.

[7] S. Felsner and S. Kappes. Orthogonal surfaces and their CP-orders. Order, vol. 25, no 1, p. 19-47, 2008.

[8] R. Forman, Witten–Morse theory for cell complexes. Topology, vol. 37, no 5, p. 945-979, 1998.

[9] D. Gonçalves and L. Isenmann, Dushnik-Miller dimension of TD-Delaunay complexes, *Proc. of EuroCG 17*, 2017.

[10] D. Kozlov. Combinatorial algebraic topology. Springer Science and Business Media, 2007.

[11] E. Miller and B. Sturmfels, Combinatorial Commutative Algebra, Graduate Texts in Mathematics, Springer-Verlag, 2004.

[12] P. Ossona de Mendez, Geometric Realization of Simplicial Complexes, *Proc. of Int. Symp. on Graph Drawing 1999 LNCS* 1731, 323-332, 1999.

[13] H. Scarf, The Computation of Economic Equilibria, vol. 24 of Cowles Foundation Monograph, Yale University Press, 1973.

[14] W. Schnyder, Planar graphs and poset dimension, *Order* 5(4): 323-343, 1989.

[15] M. Tancer, Recognition of collapsible complexes is NP-complete, Discrete and Computational Geometry, 55(1), 21-38, 2016.

[16] W. T. Trotter. Combinatorics and partially ordered sets: Dimension theory. *Johns Hopkins Series in the Mathematical Sciences*, The Johns Hopkins University Press, 1992.

[17] I. A. Volodin, V. E. Kuznetsov and A. T. Fomenko. The problem of discriminating algorithmically the standard three-dimensional sphere. Russian Mathematical Surveys, 29(5), 71-172, 1974.

[18] J. H. C. Whitehead, Simplicial Spaces, Nuclei and m-groups. Proceedings of the London mathematical society, 2(1), 243-327, 1939.

[19] E. C. Zeeman, On the dunce hat. Topology, vol. 2, no 4, p. 341-358, 1963.