Spanning trees of dense digraphs

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Abstract

An oriented tree T of order n is *unavoidable* if every tournament of the same order contains a copy of T. We find a sufficient condition for T to be unavoidable, and use this to prove that almost all labelled oriented trees are unavoidable. This settles a 28-year old conjecture of Bender and Wormald, and yields a partial result for a conjecture of Havet and Thomassé. Moreover, we prove that for all $C > 0$ every tournament of order $n + o(n)$ contains every oriented tree T of order n and maximum total degree $\Delta(T) \leq (\log n)^C$, improving a result of Kühn, Mycroft and Osthus. We discuss a related question and some work in progress.

1 Unavoidable and almost unavoidable trees

An oriented graph H of order n is *unavoidable* if every tournament of order n contains a copy of H; otherwise H is *avoidable*. Since there exist tournaments without directed cycles, any unavoidable oriented graph must be acyclic. It is then natural to ask which trees are unavoidable.

A classical result of Rédei [20] states that every directed path is unavoidable. More recently, Thomason [22] showed that all orientations of sufficiently long cycles are unavoidable except for those which yield directed cycles—implying that all orientations of sufficiently long paths are unavoidable. Havet and Thomassé [9] then gave a complete answer for paths: with three exceptions, every orientation of a path is unavoidable (the exceptions are antidirected paths of order 3, 5 and 7, which are not contained in the directed cycle of length 3, the regular 5-vertex tournament and the Paley tournament on 7 vertices respectively). Significant attention has also been focused on the unavoidability of claws (a claw is an oriented tree formed by identifying the initial vertices of a collection of vertex-disjoint directed paths). Indeed, Saks and Sós [21] conjectured that every claw on n vertices with maximum degree at most $n/2$ is unavoidable. Lu [14, 15] gave a counterexample to this conjecture, but in the other direction showed that every claw with maximum degree at most $3n/8$ is unavoidable. Lu, Wang and Wong [16] then extended these results by showing that every claw with maximum degree at most $19n/50$ is unavoidable, but that there exist claws with maximum degree approaching $11n/23$ which are avoidable. Finding the supremum of all $c > 0$ for which every claw with maximum degree at most cn is unavoidable remains an open problem.

Some trees are far from being unavoidable. For example, the outdirected star S on n vertices (whose edges are oriented from the central vertex to each of the $n-1$ leaves) is not contained in a regular tournament on $2n-3$ vertices, since each vertex of the latter has only $n-2$ outneighbours. That is, there exist tournaments with almost twice as many vertices as S which do not contain a copy of S. On the other hand, Bender and Wormald [1] proved that almost all oriented trees are 'almost unavoidable', in the sense that they are contained in almost all tournaments on the same number of vertices.

Definitions 1. We write $\mathcal{T}_n$ for the set of all labelled oriented trees of order n, $\mathcal{G}_n$ for the set of all labelled tournaments of order n, and write a.a.s. for "asymptotically almost surely (as $n \to \infty$)".

Theorem 2 (Bender and Wormald [1])**.** *There is $\mathcal{T}'_n \subseteq \mathcal{T}_n$ of size $(1 - o(1))|\mathcal{T}_n|$ such that if $G \in \mathcal{G}_n$ is chosen uniformly at random then a.a.s. G contains all trees in $\mathcal{T}'_n$.*

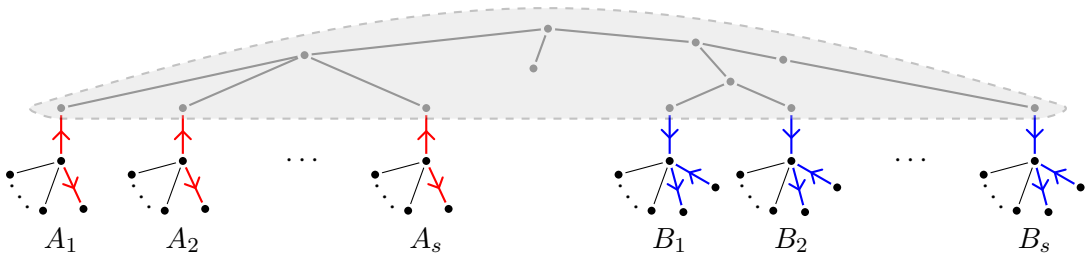


Figure 1: An α-nice tree T has $s = \lceil \alpha n \rceil$ pendant stars $A_1, \dots A_s$ which contain an out-leaf of T such that the edge between $T - A_i$ and A_i is directed away from A_i, and also s pendant stars $B_1, \dots, B_s$ which contain both an in-leaf of T and an out-leaf of T such that the edge between $T - B_i$ and B_i is directed towards B_i. In this illustration we only indicate the orientations of edges specified by this definition. The shaded area is the subtree $T - \bigcup_{i \in [s]} \big(V(A_i) \cup V(B_i)\big)$.

Hence, if $T \in \mathcal{T}_n$ and $G \in \mathcal{G}_n$ are chosen uniformly at random, then a.a.s. G contains a copy of T. In the same paper Bender and Wormald conjectured that this holds for every tournament G—that is, that almost all labelled oriented trees are unavoidable. We confirm their conjecture [18].

Theorem 3. *If $T \in \mathcal{T}_n$ is chosen uniformly at random then a.a.s. T is unavoidable.*

The following definitions are crucial for our proof of Theorem 3.

Definitions 4. We say that a subtree T' of a tree T is *pendant* if $T - T'$ is connected. Let $\alpha > 0$. An oriented tree T of order n is *α-nice* (see Figure 1) if, writing $s := \lceil \alpha n \rceil$, T contains $2s$ vertex-disjoint pendant oriented stars $A_1, \dots, A_s$ and $B_1, \dots, B_s$ and for each $i \in [s]$

- A_i is a subtree of T which contains an out-leaf of T and the edge between A_i and $T - A_i$ is oriented away from A_i, and
- B_i is a subtree of T which contains both an in-leaf of T and an out-leaf of T and the edge between B_i and $T - B_i$ is oriented towards B_i.

Most of the work involved in proving Theorem 3 lies in the proof of the following theorem, which states that large nice oriented trees with polylogarithmic maximum degree are unavoidable.

Theorem 5. *For every $\alpha, C > 0$ there exists n_0 such that if T is an α-nice tree of order $n \geq n_0$ and $\Delta(T) \leq (\log n)^C$ then T is unavoidable.*

We can choose $T \in \mathcal{T}_n$ uniformly at random by choosing a uniformly-random orientation of a uniformly-random labelled tree. So (as proved by Moon), almost all labelled trees satisfy the degree condition of Theorem 5.

Theorem 6 ([17, Corollaries 1 and 2])**.** *Choose $T \in \mathcal{T}_n$ uniformly at random. For all $\varepsilon > 0$, a.a.s.*

$$(1 - \varepsilon)\frac{\log n}{\log \log n} \leq \Delta(T) \leq (1 + \varepsilon)\frac{\log n}{\log \log n}.$$

We prove that almost all labelled oriented trees are $\frac{1}{250}$-nice. Together, theorems 5, 6 and 7 (with $C = \varepsilon = 1$ and $\alpha = \frac{1}{250}$) immediately imply Theorem 3.

Theorem 7. *If we choose $T \in \mathcal{T}_n$ uniformly at random, then a.a.s. T is $\frac{1}{250}$-nice.*

2 Beyond unavoidability—a conjecture of Havet and Thomassé

Let T be an oriented tree of order n. If T is avoidable, it is natural to ask what is the smallest integer $g(T)$ such that every tournament of order $g(T)$ contains a copy of T. With this notation, T is unavoidable if and only if $g(T) = |T| = n$. Sumner conjectured that for every oriented tree T on n vertices we have $g(T) \leq 2n - 2$, and the example of an outdirected star described above demonstrates that this bound is best possible. Kühn, Mycroft and Osthus [12, 13] used a randomised embedding algorithm to prove that Sumner's conjecture holds for sufficiently large n; previous upper bounds on $g(T)$ had been established by Chung [3], Wormald [23], Häggkvist and Thomason [6], Havet [7], Havet and Thomassé [8] and El Sahili [5]. In particular, El Sahili proved that $g(T) \leq 3n - 3$ for every oriented tree T of order n, and this remains the best general upper bound on $g(T)$ for small n. Kühn, Mycroft and Osthus [13] improved this for large oriented trees of bounded maximum degree, proving that, for all $\alpha, \Delta > 0$, if T is a sufficiently large oriented tree of order n with $\Delta(T) \leq \Delta$, then $g(T) \leq (1 + \alpha)n$. In other words, bounded degree trees are close to being unavoidable, in the sense that they are contained in every tournament of slightly larger order.

Our proof of Theorem 5 makes use of the aforementioned random embedding algorithm of Kühn, Mycroft and Osthus, using somewhat sharper estimates on certain quantities associated with the random embedding. In particular, using these stronger estimates we are able to establish the same bound on $g(T)$ for oriented trees whose maximum degree is at most polylogarithmic in n (rather than bounded by a constant as above). This is the following theorem, which we use repeatedly in the proof of Theorem 5, and which may be of independent interest.

Theorem 8. *For all $\alpha, C > 0$ there exists n_0 such that if T is an oriented tree of order $n \geq n_0$ with $\Delta(T) \leq (\log n)^C$ and G is a tournament of order at least $(1 + \alpha)n$, then G contains T.*

Let T be a tree of order n with $\ell = \ell(T)$ leaves. Havet and Thomassé [7] conjectured that $g(T) \leq n + \ell - 1$, and Havet [7] has proved that $g(T) \leq n + \frac{3}{2}(\ell^2 - 3\ell) + 5$. The conjecture is known to hold for paths and it is easy to see that it holds for all stars and unavoidable trees; hence it holds for almost every tree by Theorem 3. While the general statement remains open, we note that for all $\alpha, C > 0$ Theorem 8 confirms their conjecture for every sufficiently large tree T such that $\Delta(T) \leq \big(\log n\big)^C$ and $\ell(T) \geq \alpha n$. (Note that some trees satisfy the conditions of Theorem 8 but are not α-nice, so Theorem 8 is slightly better than Theorem 3 with respect to Havet and Thomassé's conjecture.)

3 An extension—trees in dense digraphs

A well-known result of Komlós, Sárközy and Szemerédi [10] states that for all $\varepsilon, \Delta > 0$ there exists n_0 such that every graph of order $n \geq n_0$ and minimum degree $\big(\frac{1}{2} + \varepsilon\big)n$ contains every (spanning) tree of order n and maximum degree Δ. (This has subsequently been improved by the same and other authors in many directions [2, 4, 11].) The theorems above suggest that similar approach could be used to establish a digraph analogue of their result.

However, our proofs of theorems 3, 5 and 8 use the fact that the host graph is a tournament in several places. For instance, we rely on a characterisation of the structure of large tournaments and a randomised embedding algorithm (from [13]), both of which would need to be adapted. We discuss progress in this direction.

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