On conservative graphs and Skolem systems

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Abstract

The *conservative number* of a graph G is the minimum positive integer M, such that G admits an orientation and a labeling of its edges by distinct integers in $\{1, 2, \ldots, M\}$, such that at each vertex of degree at least three, the sum of the labels on the in-coming edges is equal to the sum of the labels on the out-going edges. A graph G is *conservative* if $M = |E(G)|$.

Let $m_1, m_2, \ldots, m_n$ be positive integers with $m_i \geq 3$ for $1 \leq i \leq n$, $M = \sum_{i=1}^{n} m_i$, and $t \in \{0, 1\}$. Given a family of n disjoint sequences of integers $S = \{D_1, D_2, \ldots, D_n\}$, we say that S is an $(m_1, m_2, \ldots, m_n; k; t)$*-Skolem system of order* n *and size* M if it satisfies the following conditions:

1. $|D_i| = m_i$, for $1 \leq i \leq n$,
2. $\sum_{d \in D_i} d = 0$, for $1 \leq i \leq n$, and
3. $\mathrm{abs}(S)$ is a partition of $[1, r] \cup [r+1+k, M+k-1] \cup \{M+k+t\}$, for a positive integer r,

where $\mathrm{abs}(S)$ denotes $\{\mathrm{abs}(D_1), \mathrm{abs}(D_2), \ldots, \mathrm{abs}(D_n)\}$ and for $D \in S$, $\mathrm{abs}(D)$ is the set of the absolute values of the elements in D.

In this talk we will show, by means of Skolem systems, that the conservative number of a galaxy (a disjoint union of stars) of size M is M for $M \equiv 0, 3 \pmod 4$, and $M+1$ otherwise.

Bryant, Gavlas and Ling introduced Skolem-type and hooked Skolem-type m-cycle difference sets, for an integer $m \geq 3$ (see Definition 1.1 in [2]). Based on those ideas, we proposed the following structure in [3]:

Definition 1. *Let $m_1, m_2, \ldots, m_n$ be positive integers with $m_i \geq 3$ for $1 \leq i \leq n$, $M = \sum_{i=1}^{n} m_i$, k be a non-negative integer, and $t \in \{0, 1\}$.*

Given a family of n disjoint sequences of integers $S = \{D_1, D_2, \ldots, D_n\}$, we say that S is an $(m_1, m_2, \ldots, m_n; k; t)$-Skolem system of order n and size M *if it satisfies the following conditions*

1. *$|D_i| = m_i$ for $1 \leq i \leq n$,*
2. *$\sum_{d \in D_i} d = 0$ for $1 \leq i \leq n$, and*
3. *$\mathrm{abs}(S)$ is a partition of $[1, r] \cup [r+1+k, M+k-1] \cup \{M+k+t\}$, for a positive integer r,*

where $\mathrm{abs}(S)$ *denotes* $\{\mathrm{abs}(D_1), \mathrm{abs}(D_2), \ldots, \mathrm{abs}(D_n)\}$ *and for $D \in S$,* $\mathrm{abs}(D)$ *is the set of the absolute values of the elements in D.*

When $m_1 = m_2 = \cdots = m_n = m$, we write $(mn; k; t)$-Skolem system.

In 2003, Bryant, Gavlas and Ling used Skolem and Langford type sequences in order to provide the following result.

Theorem 1 (Bryant, Gavlas and Ling [2])**.** *Let m and n be positive integers with $m \geq 3$, and $t \in \{0, 1\}$. There exists a sequenceable $(mn; 0; t)$-Skolem system if and only if $mn \equiv t, 3-t \pmod 4$.*

We provided the following generalization:

Theorem 2 ([3]). *Let $m_1, m_2, \ldots, m_n$ be positive integers with $m_i \geq 3$ for $1 \leq i \leq n$, $M = \sum_{i=1}^{n} m_i$, and $t \in \{0, 1\}$. There exists a $(m_1, m_2, \ldots, m_n; 0; t)$-Skolem system of order n and size M if and only if $M \equiv t, 3 - t \pmod 4$.*

As a consequence of a result by Archdeacon, Boothby and Dinitz in [1], we have that

Theorem 3. *The complete bipartite graph $K_{m,n}$ is conservative if and only if*

$$mn \equiv 0, 3 \pmod 4.$$

Graphs of the form $K_{1,r}$ with $r \geq 3$ are called *stars*. A *galaxy* is a disjoint union of stars. Given positive integers $m_1, m_2, \ldots, m_n$ with $m_i \geq 3$ for $1 \leq i \leq n$, we denote by $G(m_1, m_2, \ldots, m_n)$ the galaxy with n stars of size $m_1, m_2, \ldots, m_n$. If $m_1 = m_2 = \cdots = m_n = m$, then we write $G_{m,n}$.

For a bipartite graph G with partite sets $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ such that $d_G(x_i) = p_i \geq 3$ and $d_G(y_i) = q_i \geq 3$, if G is conservative then both $G(p_1, p_2, \ldots, p_m)$ and $G(q_1, q_2, \ldots, q_n)$ are conservative. Moreover, Theorem 3 implies that $G_{m,n}$ and $G_{n,m}$ are conservative if and only if $mn \equiv 0, 3 \pmod 4$.

By means of Skolem systems, we have the following result:

Theorem 4 ([3]). *Let $m_1, m_2, \ldots, m_n$ be positive integers with $m_i \geq 3$, $G(m_1, m_2, \ldots, m_n)$ is conservative if and only if there exists an $(m_1, m_2, \ldots, m_n; 0; 0)$-Skolem system. Furthermore, an $(m_1, m_2, \ldots, m_n; 0; 1)$-Skolem system represents an $(\sum_{i=1}^{n} m_i + 1)$-directed numbering of $G(m_1, m_2, \ldots, m_n)$.*

References

[1] D.S. Archdeacon, T. Boothby and J.H. Dinitz, Tight Heffter arrays exist for all possible values. *J. Combin. Des.* **25** (2017), no. 1, 5–35.

[2] D. Bryant, H. Gavlas, and A.C.H. Ling, Skolem-type Difference Sets for Cycle Systems, *Electron. J. Combin.* **10** (2003) # R38.

[3] I.A. Goldfeder and J. Tey, A note on conservative galaxies, Skolem systems, cyclic cycle decompositions, and Heffter arrays. *(Submitted.)*