List strong edge coloring of a $(2,3)$-bipartite graph

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Abstract

A strong edge coloring of a graph is a proper edge coloring such that each color class is an induced matching. The strong chromatic index $\chi'_s(G)$ of a graph G is the minimum number of colors for G to have a strong edge coloring. A bipartite graph is called an (a,b)-bipartite graph if one part has maximum degree at most a and the other part has maximum degree b. Brualdi and Quinn Massey (1993) conjectured that for a (a,b)-bipartite graph G, $\chi'_s(G) \leq ab$, as a refinement of the conjecture by Faudree, Gyárfás, Schelp, Tuza (1990). The conjecture holds for $a \in \{2,3\}$, by Naprasit (2014) and Huang, Yu, Zhou (2017).

In this talk, we study list strong edge coloring, and denote the list strong chromatic index by $ch'_s(G)$. We show that for any (2,3)-bipartite graph G, $ch'_s(G) \leq 6$. From a relation to incidence coloring, this is also a generalization of a result by Benmedjdoub et. al. (2017), on incidence 6-choosability of Hamiltonian cubic graph.

1 Introduction and the main result

In this paper, we always consider a loopless graph. For a graph allowed to have a multiple edge, we call it a *multigraph*. In addition, we call a simple graph which is a graph without multiple edge, just a *graph*.

A *strong edge coloring* of a graph is an edge coloring such that each color class is an induced matching, and the *strong chromatic index* $\chi_s(G)$ is the smallest integer k such that G allows a strong edge coloring using k colors. There are intensive research on this topic started from one famous conjecture of Erdős and Nešetřil (1989) on strong edge chromatic index. A bipartite version of the conjecture was suggest by Faudree, Gyárfás, Schelp and Tuza in [7], and Brualdi and Quinn Massey refined the conjecture as follows. A (a,b)-bipartite graph G means a bipartite graph with a bipartition (A,B) such that $\max\{\deg_G(v) \mid v \in A\} \leq a$ and $\max\{\deg_G(v) \mid v \in B\} \leq b$.

Conjecture 1 ([6]). *For every (a,b)-bipartite graph G, $\chi_s(G) \leq ab$.*

We remark that the conjecture is still open, and see [2, 5, 9, 12, 13] for some recent partial results. In [12], it was shown that a (2,Δ)-bipartite graph satisfies the conjecture.

Brualdi and Quinn Massey [6] introduced the notion of an incidence coloring, in a relation to a strong edge coloring of a special bipartite graph. An *incidence* of a multigraph is a pair (u,e) of a vertex u and an edge e of G such that u is an endpoint of e. In this case, we say (u,e) is an incidence on the edge e. We denoted by $I(G)$ all incidences of a multigraph G. Two incidences (u,e) and (v,f) are *adjacent* if one of the following holds: (i) $u = v$; (ii) $e = f$; (iii) $uv = e$ or $uv = f$. An *incidence coloring* of a multigraph G is a function φ from $I(G)$ to the set of colors so that $\varphi(u,e) \neq \varphi(v,f)$ for any two adjacent incidences (u,e) and (v,f). An *incidence k-coloring* means an incidence coloring using at most k colors. The *incidence chromatic number* $\chi_i(G)$ of a multigraph G is the smallest integer k such that G admits an incidence k-coloring. It was known that an incidence coloring of a multigraph G corresponds to strong edge coloring of the *subdivision* $S(G)$ of G, where $S(G)$ is the graph obtained from G by subdividing every edge of G. See [6] for more details. Note that the subdivision of a graph with maximum degree Δ is a $(2,\Delta)$-bipartite

graph. In this context, Brualdi and Quinn Massey introduced the notion of incidence coloring and then suggested Conjecture 1.

An incidence coloring is also interpreted as a special case of directed star coloring. For a multigraph G, let $D(G)$ be the digraph obtained from G by replacing each edge of G by two opposite arcs. If we associate each incidence (v, vw) of $I(G)$ with the arc (v, w) of $D(G)$, then an incidence coloring of G is equal to a directed star coloring of $D(G)$. A *directed star coloring*, introduced by Algor and Alon in [1], is a coloring of the arcs of a digraph D so that each color class induces a forest of directed stars whose arcs are directed towards the center, and the *directed star arboricity* of D is the minimum number of colors to obtain a directed star coloring of G. See [8] for more details.

It is easy to observe that for any connected graph G with at least three vertices, $\Delta(G) + 1 \leq \chi_i(G) \leq 3\Delta(G) - 2$, and in [6], the authors conjectured that any simple graph G satisfies that $\chi_i(G) \leq \Delta(G)+2$. This conjecture was called the incidence coloring conjecture, but few years later, counterexamples were found in [8]. Even though it turned out be false, it leads several interesting questions about graphs satisfying the inequality. On the class of cubic graphs, in [3, 14], it was shown that some several classes of cubic graphs are incidence 5-colorable. In 2005, Maydanskiy [11] showed that any cubic graph is incidence 5-colorable:

Theorem 1 ([11]). *For every cubic graph G, $\chi_i(G) \leq 5$.*

It is known to be NP-hard to determine if the incidence chromatic number of a cubic graph is 4 (see [10]).

In this paper, we study a list version of incidence coloring, and this automatically gives a result in strong edge choosability of a bipartite graph. For a multigraph G, an *incidence list assignment* L of G is a function defined on $I(G)$ such that L associates each $(v, e) \in I(G)$ with a nonempty set $L(v, e)$. If $|L(v, e)| \geq k$ for each $(v, e) \in I(G)$, then we say it is an *incidence k-list assignment*. When an incidence list assignment L of G is given, we say G is *incidence L-choosable* if there is an incidence coloring φ such that $\varphi(v, e) \in L(v, e)$ for each incidence (v, e) of G. A multigraph G is said to be incidence k-choosable if G is incidence L-choosable for any incidence k-list assignment. Then *incidence choice number* $ch_i(G)$ is the smallest integer k such that G is incidence k-choosable.

As we noted in Theorem 1, the incidence chromatic number for a cubic graph is known to be at most 5, the incidence choice number of a Hamiltonian cubic graph is studied the most recently in [4] as follows.

Theorem 2 ([4]). *For every Hamiltonian cubic graph G, $ch_i(G) \leq 6$.*

In this paper, we generalize Theorem 2, as follows:

Theorem 3 (Main result). *For a subcubic loopless multigraph G, $ch_i(G) \leq 6$.*

By a relation to strong edge coloring, we immediately obtain the following. The *list strong chromatic index* $ch'_s(G)$ of a graph G is the smallest integer k such that G is strong edge L-choosable for every list assignment L on $E(G)$ satisfying that $|L(e)| \geq k$ for any $e \in E(G)$. Then Theorem 3 is equivalent to the following:

Theorem 4 (Equivalent to Theorem 3). *For every $(2, 3)$-bipartite graph G, $ch'_s(G) \leq 6$.*

Note that the bound 6 is tight by a complete bipartite graph $K_{2,3}$.

2 Outline of proof of Theorem 3

We prove the following Theorems 5 and 6 first, and we skip their proofs here.

Theorem 5. *For a 2-connected cubic graph G, $ch_i(G) \leq 6$.*

For a graph G and an incidence list assignment L of G, an edge $e = uv$ of G is said to be *freely L-choosable* if for any two distinct colors $c_1 \in L(u, e)$ and $c_2 \in L(v, e)$, there is an incidence L-coloring φ such that $\varphi(u, e) = c_1$ and $\varphi(u, e) = c_2$.

Theorem 6. *Let G be a semicubic graph with at least one pendant vertex, L be an incidence 6-list assignment. Suppose that the graph obtained from G by deleting all pendant vertices of G is K_1 or 2-connected. Then for every pendant edge $e^* = uv$ of G, e^* is freely L-choosable.*

We mention one simple fact in graph theory.

Fact 1. *For a cubic graph G, G has a cut-vertex if and only if it has a cut-edge.*

Here, we give a proof of the main result, using Theorems 5 and 6.

Sketch of the proof of Theorem 3. Let G be a minimum counterexample with respect to the number of vertices. Then there is an incidence 6-list assignment such that G is not incidence L-choosable.

Note that G is connected. First, we will show the following first and we skip the proof here.

Claim 1. *The graph G is cubic and has no multiple edge.*

By Claim 1, G is simple and cubic. If G is 2-connected, then it holds by Theorem 5. Suppose that G is not 2-connected. Then by Fact 1, G has cut-edges, and let K be the set of all cut-edges of G (note that $K \neq \emptyset$). Let $G_1, \ldots, G_m$ be the components of $G - K$, which is the graph obtained from G by deleting the edges in K. Note that each G_i is K_1 or 2-connected. For each $i \in [m]$, we let G_i^* be the graph obtained from G_i by adding all edges in K incident to a vertex of G_i.

Let $\mathcal{G}$ be the graph whose vertices are $\{G_1, \ldots, G_m\}$, and G_i and G_j are adjacent if and only if there is a cut edge between a vertex of G_i and a vertex of G_j. Then it is easy to see that $\mathcal{G}$ is a tree and the number of edges in G (actually in K) between G_i and G_j is at most one. Thus we can relabel the vertices of $\mathcal{G}$ so that for each $i \in \{2, \ldots, m\}$, G_i is adjacent to exactly one G_j for some $j \in [i-1]$, and so we let e_i^* be the edge of G_i^* between G_i and such G_j.

Note that each G_i^* is semi-cubic at least one pendant edge, and the graph obtained from G_i^* by deleting all pendant vertices is G_i, which is K_1 or 2-connected. Thus we can apply Theorem 6 to each G_i^*. By Theorem 6, G_1^* has an incidence L-coloring φ. Suppose that $G_1^* \cup \cdots \cup G_i^*$ has an incidence L-coloring φ for some $i \in [m]$. Then among the incidences of G_{i+1}^*, only two incidence on the edge e_{i+1}^* are precolored under φ. By applying Theorem 6 to G_{i+1}^* and e_{i+1}^*, it follows that φ is well-extended to an incidence L-coloring of $G_1^* \cup \cdots \cup G_i^* \cup G_{i+1}^*$. Hence, by repeating the arguments, we obtain an incidence L-coloring of G. □

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