Connected greedy colouring in claw-free graphs

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Abstract

An ordering of the vertices of a graph is *connected* if every vertex (but the first) has a neighbor among its predecessors. The greedy colouring algorithm of a graph with a connected order consists in taking the vertices in order, and assigning to each vertex the smallest available colour. A graph G is *good* if for every connected induced subgraph H of G and for every connected order $\mathcal{O}$ of H, the greedy algorithm gives H an optimal colouring. We give the characterization of good claw-free graphs in terms of minimal forbidden induced subgraphs.

A *k-colouring* for a graph G is any function π from $V(G)$ to $\{1, \ldots, k\}$ such that for any edge $uv \in E(G)$, $\pi(u) \neq \pi(v)$. The smallest integer k such that G admits a k-colouring is called the *chromatic number* of G and is denoted by $\chi(G)$. A $\chi(G)$-colouring of G is called an *optimal colouring of* G. Computing the chromatic number is known to be NP-hard.

Let G be a graph and $\mathcal{O} = [v_1, \ldots, v_n]$ be a linear ordering of its vertices. The *greedy colouring algorithm* (greedy algorithm for short) applied to $(G, \mathcal{O})$ consists in taking the vertices in the order $\mathcal{O}$, and giving to each vertex a colour equal to the smallest positive integer not used by its neighbours already coloured. This obviously produces a colouring.

For every graph, there exists an order $\mathcal{O}$ for the vertices such that the greedy algorithm produces an optimal colouring. To see this, consider an optimal colouring π, and consider the following ordering: first take vertices with colour 1, then vertices with colour 2, and so on. But this method has no practical interest to compute optimal colourings, since to find the ordering, an optimal colouring has to be known.

It is also well known that for some graphs, there exist orderings that produce colourings very far from the optimal, for instance consider two disjoint sets on n vertices, say $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. Add all possible edges between A and B, except edges $a_i b_i$, $i \in \{1, \ldots, n\}$. This produces a bipartite graph G. However, the greedy algorithm applied to the order $[a_1, b_1, a_2, b_2, \ldots, a_n, b_n]$ produces a colouring with n colours.

Let P_k denote the path on k vertices. When H and G are graphs, we say that G is *H-free* if G does not contain an induced subgraph isomorphic to H. The graph such that the greedy algorithm on every order gives every induced subgraph of it an optimal colouring are fully characterized.

Theorem 1 (see [1, 2])**.** *For every graph G, the following properties are equivalent.*

- *G is P_4-free.*
- *For every induced subgraph H of G and every linear order $\mathcal{O}$ of $V(H)$, the greedy colouring algorithms applied to $(H, \mathcal{O})$ produces an optimal colouring of H.*

There are many ways to order the vertices of a graph with the hope to obtain a better colouring. In this paper, we focus on *connected* orders. An order $\mathcal{O} = [v_1, \ldots, v_n]$ for a graph G is *connected* if for every $2 \leq i \leq n$, there exists $j < i$ such that $v_j v_i \in E(G)$. A connected order exists if and only if G is connected, and is efficiently produced by search algorithms such as BFS, DFS (or more simply by the algorithm *generic search*). We say that a graph G is *good* if for every connected induced subgraph H of G and every connected order $\mathcal{O}$ of H, the greedy algorithm produces an

optimal colouring of H. Also, a connected order $\mathcal{O}$ of a graph G is *good* if it produces an optimal colouring of G. A graph or a connected order is *bad* if it is not good. A graph is *minimally bad* if it is bad and all other connected induced subgraphs of it are good. Connected orders are better than general orders for colouring bipartite graphs.

Theorem 2 (see [3])**.** *Every bipartite graph is good.*

However, unlike general orders, it is not true that for every graph, there exists a connected order that provides an optimal colouring, see [4] for example. A similar claw-free example is given here:

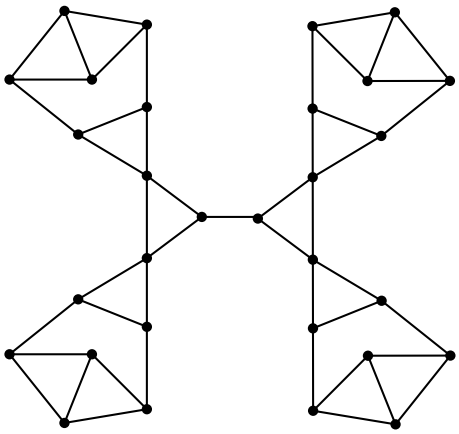


Figure 1: A claw-free graph where every connected order is bad.

The connected greedy colouring has recently been studied. In [3], they define $\Gamma_c(G)$ as the maximum number k such that there exists a connected order producing a k-colouring of G. They also proved that checking if $\Gamma_c(G) \geq k$ is NP-hard if k is a part of the input. In [5], they show that this problem remains NP-hard even when $k = 7$. A graph G is good in our definition if for every connected induced subgraph H of G, $\Gamma_c(H) = \chi(H)$. Note that their results imply also that checking if there exists a bad connected order for a graph is NP-hard, but do not imply NP-hardness on recognizing good graphs (since a class of good graphs is hereditary by our definition). The complexity of recognizing good graphs remains open. In [4], they gave several examples of small graphs that are not friendly with connected orders. In [6], they defined a more restricted good graph with respect to connected orders and gave the complete characterization of this class. Therefore, their class is also good by our definition.

However, the list of excluded induced subgraphs for the class of good graphs is still unknown. Equivently, no description of minimally bad graphs is known. Our goal is to prove an analogue of Theorem 1 for connected orders. If we restrict our attention to claw-free graphs, we are able to give this description (where the *claw* is the graph on $\{a, b, c, d\}$ with edges ab, ac and ad). This is our main result that we now state precisely.

A *cycle* in G is a sequence of distinct vertices $v_1 \dots v_k$ such that $v_i v_{i+1} \in E(G)$ for $i \in \{1, \dots, k\}$ (the index is taken modulo k). The edges $v_i v_{i+1}$ are the edges of the cycles, the other edges between the vertices of the cycle are called its *chords*. The length of a cycle is the number of its edges (here k). A *hole* is a cycle of length at least 4 that has no chord. A *path* in G is a sequence $P = v_1 \dots v_k$ of distinct vertices of G such that $v_i v_j \in E(G)$ if and only if $|i - j| = 1$ (paths are often refered

to as induced paths or chordless paths). Vertices v_1 and v_k are the *ends* of P and the rest of the vertices are *internal*. The length of a path is the number of its edges. A hole (cycle, path) is *even* or *odd* according to the parity of its length. A path in a graph G is *flat* if all its internal vertices are of degree 2 (in G). A *triangle* is a graph on three vertices and they are all adjacent.

A graph H is a *prism* if:

- $V(H) = V(Q_1) \cup V(Q_2) \cup V(Q_3)$.
- For $i \in \{1, 2, 3\}$, Q_i is a path of length ≥ 1 with two ends a_i and b_i.
- Q_1, Q_2 and Q_3 are vertex-disjoint.
- $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are two triangles.
- These are the only edges in H.

A prism is *parity* if its three paths have the same parity and is *imparity* otherwise. Note that a prism contains an odd hole if and only if it is imparity. A parity prism is *even* (*odd*) if the lengths of its three paths are even (odd).

We call *obstructions* the graphs represented in Figure 2 with the following additional specifications:

- The orientation represented for each graph has no special meaning. It is an indication of how a bad connected order can be found for it. The orientation does not fully specify this order. The arrow should be seen from a small to a big vertex with respect to this order. The chromatic number of each graph is 3 and the last vertex in every bad order receives colour 4.
- All the straight lines are edges, all the curved lines are paths of length ≥ 1.
- The hole in F_1 is odd.
- The only curved line in F_2 is of length ≥ 1. The orientation of the only unoriented edge depends on the parity of this curved line.
- The only curved line in F_3 is of length ≥ 1.
- The hole in F_5 is even.
- All curved lines in F_7, F_8, F_9, F_{10} are of length ≥ 2.
- F_7 is an imparity prism. The lower path is of different parity from the other two paths.
- The prism in F_8 is an even prism. The upper path of the prism contains two curved lines: the first one is odd, the second is even.
- The prisms in F_9 and F_{10} are odd prisms.
- The upper path of the prism in F_9 contains two curved lines of odd length.
- The upper and lower paths of the prism in F_{10} contain four curved lines of even length.
- The length of the only long cycle in F_{11} is odd ≥ 3.

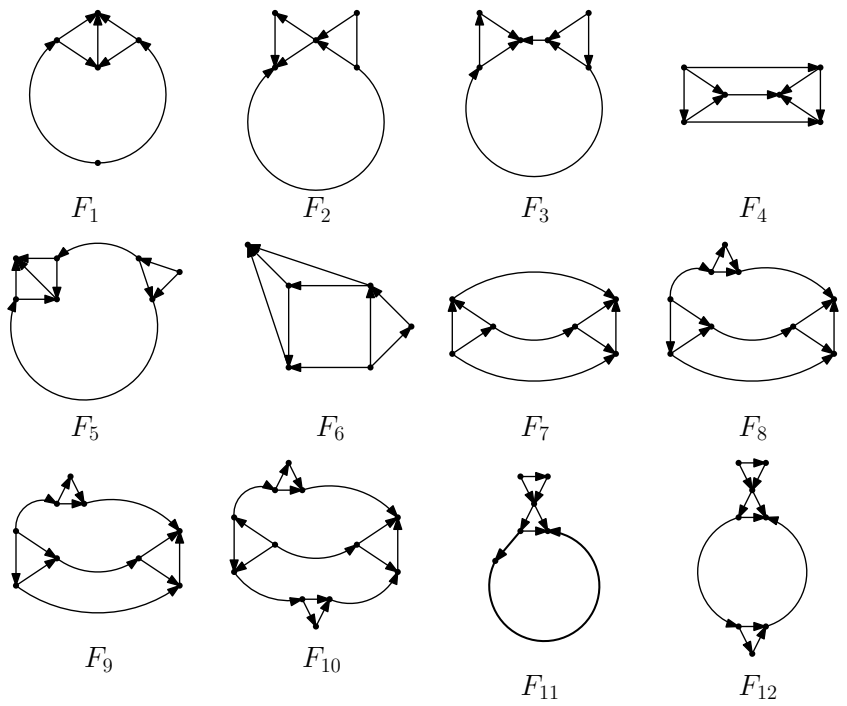


Figure 2: List of obstructions

- The length of two curved lines in F_{12} is odd ≥ 3.

Our main result is the following.

Theorem 3. *Let G be a claw-free graph. Then G is good if and only if G does not contain any obstruction as an induced subgraph. Equivalently, a claw-free graph is minimally bad if and only if it is an obstruction.*

References

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