Chains and anti-chains of length four in partially ordered sets

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Abstract

We show that the minimum density of homogeneous subsets (chains or anti-chains) of size four in partially ordered sets (posets) is asymptotically 1/27, which is known to be best possible. This verifies the first non-trivial case of a question of Samotij and Sudakov.

Moreover, we obtain a stability version of this result stating that any large poset with a density of homogeneous subsets of almost 1/27 is close to either the union of three linear orders of roughly equal size, or, the ordering of three sets of equal size, with no relations inside the sets.

The main ingredient used in the proof is the framework of flag algebras. Our work extends an earlier result of Balogh et al. on the analogous question for monotone subsequences of length four in permutations. The main difficulty of our generalization was to fully understand the set of all the extremal and nearly extremal constraints that a flag algebraic proof must respect, which is more complex in the poset setting.

The theorem of Erdős and Szekeres [5] shows that if $n, k \in \mathbb{N}$ such that with $n \geq k^2 + 1$, then every permutation on $[n] := \{1, \ldots, n\}$ contains at least one increasing or decreasing subsequence of length $k+1$. A similar statement holds for posets $(X, \succeq)$ with $|X| = n \geq k^2 + 1$, as here we find that this poset contains at least one chain or anti-chain of length $k+1$. This is an easy consequence of Dilworth's theorem [3], or, its dual version [4], which is in fact much simpler to deduce than Dilworth's theorem itself. This implies that for $n \gg k^2$, we see many chains and/or anti-chains of length $k+1$. In the case of permutations, the question of how many monotone subsequences of length $k+1$ must a permutation on $[n]$ contain was, according to Myers [7], first posed by Atkinson, Albert and Holton. Myers [7] described permutations

$$\begin{aligned}\tau_k(n) = (&t_{k-1}+1, \ \ldots, \ n,\\ &t_{k-2}+1, \ \ldots, \ t_{k-1},\\ &\vdots\\ &1 \quad , \ldots, \quad t_1),\end{aligned}$$

where every $t_j \in \left\{ \left\lfloor \frac{jn}{k} \right\rfloor, \left\lceil \frac{jn}{k} \right\rceil \right\}$, i.e. a subdivision into k monotonously increasing subsequences with size differing by at most 1, with every monotonously increasing sequence contained in one of these; see also Figure 1 for a drawing of one of such a permutation. This yields

$$m_k(\tau_k(n)) = r \cdot \binom{\lceil n/k \rceil}{k+1} + (k-r) \cdot \binom{\lfloor n/k \rfloor}{k+1} \approx \frac{1}{k^k}\binom{n}{k+1}$$

monotone subsequences of length $k+1$, where $r \equiv n \mod k$.

Myers further observed that this construction is minimal in the case $k = 2$, which is a direct consequence of the so-called Goodman's formula [6] stating that any graph contains at least $m_2(\tau_2(n))$ triangles or sets of three independent vertices. He conjectured that the permutations described above are extremal for all values of n and k:

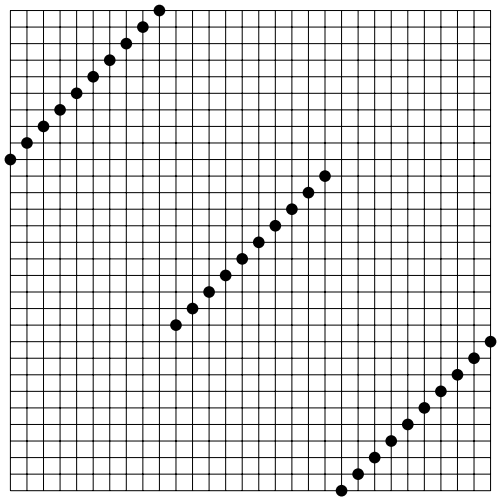


Figure 1: The permutation $\tau_3(30)$.

Conjecture 1 ([7]). *Let $n, k \in \mathbb{N}$. Every permutation on $[n]$ has are at least $m_k(\tau_k(n))$ monotone subsequences of length $k+1$.*

the case $k = 3$ was verified for n sufficiently large by Balogh et al. [1] using the flag algebra framework of Razborov [8]. Samotij and Sudakov [9] showed that this is also the case when $n \leq k^2 + ck^{3/2}/\log(k)$ for some absolute constant $c > 0$ and k large enough. Their work uses the language of posets, where monotone sequences become chains and anti-chains, albeit only proving the case for posets of order dimension at most two which coincides with the posets derived from permutations; see [4]. However, they asked whether Myers' conjectured bound also holds for homogeneous sets, i.e., chains and anti-chains, in an arbitrary poset.

Question 1 ([9]). *Let $n, k \in \mathbb{N}$. Is it true that at least one of the posets on $[n]$ that minimizes the number of homogeneous sets of size k has dimension at most two?*

Indeed, the answer is yes in the case $k = 2$, which readily follows from Goodman's formula.

In the permutation setting, Myers also offered a weaker conjecture asking only about the asymptotics of the minimum number of monotone subsequences.

Conjecture 2 ([7]). *Let $k \in \mathbb{N}$ and $n \to \infty$. In any permutation of $[n]$ there are at least*

$$(1 - o(1)) \cdot \binom{n}{k+1} \Big/ k^k$$

monotone subsequences of length k+1.

Our main result is an asymptotic solution of the question of Samotij and Sudakov for $k = 3$, which generalizes the result of Balogh et al. [1] to posets.

Theorem 1. *For any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ s.t. for every $n \geq n_0$ the following is true: Every poset $(X, \succeq)$ on $[n]$ contains at least $(1 - \epsilon)\binom{n}{4}/27$ homogeneous sets of size four.*

As we already mentioned, the main ingredient in our proof of Theorem 1 is the flag algebra method. Specifically, we first translate the problem to the language of *comparability graphs*. A comparability graph of a poset $(X, \succeq)$ is an undirected graph $(X, E_{\succeq})$ where $\{x, y\} \in E_{\succeq} \iff x \succeq y \vee y \succeq x$. Clearly, there are one-to-one correspondences between chains / anti-chains in a poset and cliques / stable sets in the corresponding comparability graph, respectively. Then, we use a semidefinite relaxation to the question whether the proportion of homogeneous vertex-subsets of size 4 in every comparability graph is at least $1/27 - o(1)$, to which, with a computer assistance (using SAGE [10] and CSDP [2]), we find an exact solution.

Moreover, a closer look to the structure of the solution of the semidefinite program, we obtain further information about all asymptotic extremal comparability graphs. This information is then used in order to obtain the following stability-type result:

Theorem 2. *For every $\varepsilon_{\text{STAB}} > 0$ there exist $n_0 \in \mathbb{N}$ and $\delta_{\text{STAB}} > 0$ s.t. for every $n \geq n_0$ the following is true: If a comparability graph G on n vertices has less than $\left(\frac{1}{27} - \delta_{\text{STAB}}\right) \cdot \binom{n}{4}$ homogeneous vertex-subsets of size 4, then, after changing at most $\varepsilon_{\text{STAB}} \cdot n^2$ edges, G or its complement is isomorphic to the n-vertex graph given by three disjoint cliques with sizes differing by at most one.*

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