Balanceable graphs

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Abstract

Given a graph G and a 2-coloring $f : E(K_n) \to \{red, blue\}$, we say that f induces a *balanced* copy of G if there is a copy of G in K_n which, under coloring f, has $\lceil e(G)/2 \rceil$ red edges and $\lfloor e(G)/2 \rfloor$ blue edges or vice versa. Graphs G for which, for sufficiently large n, every coloring $f : E(K_n) \to \{red, blue\}$ with sufficiently many red and blue edges induces a balanced copy of G are called *balanceable*. In this work, we present a characterization of balanceable graphs, and analyze the balanceability of certain graph families as well as the extremal structures of the colorings that prevent to find the desired balanced graph.

1 Introduction

This work is based on the results of Zero-sum Ramsey theory flavour obtained in [2] and [3]. See also [1, 4] for closely related literature.

We will consider 2-colorings on the set of edges $E(K_n)$ of the complete graph K_n, written as follows:

$$f : E(K_n) \to \{red, blue\}.$$

Given such a coloring f, let R_f and B_f be the graphs induced by, respectively, the red and the blue edges of K_n with respect to the coloring f. Let $e(G)$ denote the number of edges of a graph G.

Given a graph G and a 2-coloring $f : E(K_n) \to \{red, blue\}$, we say that f induces a *balanced* copy of G if there is a copy of G in K_n which, under coloring f, has $\lceil e(G)/2 \rceil$ red edges and $\lfloor e(G)/2 \rfloor$ blue edges or vice versa. We will study graphs for which one can always find a balanced copy of it in every 2-coloring of $E(K_n)$ fulfilling certain constraints on the number of edges of each color. For example, a balanced copy of a path of length 2 (that is, with one red and one blue edge) can always be found on a 2-coloring of $E(K_n)$ that contains at least one red edge and one blue edge. Graphs G for which, for sufficiently large n, every coloring $f : E(K_n) \to \{red, blue\}$, with sufficiently many red and blue edges, induces a balanced copy of G are called *balanceable*.

Given a balanceable graph G, we define $\mathrm{bal}(n, G)$ as the minimum integer such that any 2-coloring $f : E(K_n) \to \{red, blue\}$ with $\min\{e(R), e(B)\} > \mathrm{bal}(n, G)$ contains a balanced copy of G. Moreover, let $\mathrm{Bal}(n, G)$ be the set of extremal graphs, namely the graphs on $\mathrm{bal}(n, G)$ edges induced by one of the colors on a coloring $f : E(K_n) \to \{red, blue\}$ with $\min\{e(R), e(B)\} = \mathrm{bal}(n, G)$ but without a balanced G. We will focus our attention on the following problems.

- Characterize all balanceable graphs.
- If G is balanceable, determine the value of $\mathrm{bal}(n, G)$.
- If G is balanceable, determine the family of extremal graphs $\mathrm{Bal}(n, G)$.

We will use standard graph theoretical notation to denote particular graphs. So, as already mentioned, K_n is the complete graph on n vertices, and $K_{1,k}$, P_k and C_k will stand, respectively, for

the star, the path and the cycle on k edges. Moreover, $H_1 \cup H_2$ will represent the disjoint union of graphs H_1 and H_2. Further, given disjoint sets $X, Y \subseteq V(G)$ on a graph G, we denote by $e(X,Y)$ the set of edges between the vertices in X and the vertices in Y. Finally, for a set $W \subseteq V(G)$ we denote by $G[W]$ the subgraph of G induced by the vertices in W.

2 Characterization of balanceable graphs

For this section, we need to introduce the following notation. A 2-edge-colored graph G is said to be of *type A* if there is a partition of the vertex set $V(G) = X \cup Y$ such that all edges in X have one color and the rest of edges have the other color. A 2-edge-colored graph G is said to be of *type B* if there is a partition of the vertex set $V(G) = X \cup Y$ such that all edges from X to Y have one color and the rest of edges have the other color. If the mentioned sets satisfy $|X| = |Y|$ in case $V(G) \equiv 0 \pmod 2$ and $||X| - |Y|| = 1$ in case $V(G) \equiv 1 \pmod 2$, then we say G is *equipartite of type A* or, respectively, *equipartite of type B*.

A major result in [3] is the determination of necessary and sufficient conditions for a graph G to be balanceable. The following theorem contains the essence of what is needed for the characterization.

Theorem 1. *Let t be a positive integer. Then, for n sufficiently large, there is an integer $\varphi(n,t)$ such that any coloring $f : E(K_n) \to \{red, blue\}$ with $\min\{e(R_f), e(B_f)\} \geq \varphi(n,t)$, contains a copy of K_{2t} either equipartite of type A or equipartite of type B.*

Suppose we have a type A-coloring of K_n and let G be a balanceable graph. Since there has to exist a balanced copy of G in K_n, G admits a partition $V(G) = X \cup Y$ such that $e(X,Y) \in \{\lfloor \frac{1}{2}e(G) \rfloor, \lceil \frac{1}{2}e(G) \rceil\}$. Similarly, a type B-coloring forces the existence of a vertex set $W \subseteq V(G)$ such that $e(G[W]) \in \{\lfloor \frac{1}{2}e(G) \rfloor, \lceil \frac{1}{2}e(G) \rceil\}$.

It turns out that these conditions are also sufficient to guaranty the balanceability of a graph G. This is given in the following theorem.

Theorem 2. *A graph G is balanceable if and only if G has both a partition $V(G) = X \cup Y$ and a set of vertices $W \subseteq V(G)$ such that $e(X,Y), e(G[W]) \in \{\lfloor \frac{1}{2}e(G) \rfloor, \lceil \frac{1}{2}e(G) \rceil\}$.*

The nature of the proof of Theorem 1, which relies on the Ramsey Theorem, the Turán numbers and the Zarankiewicz numbers, prevents at the same time to get a sharp or even good upper bounds for $\mathrm{bal}(n, G)$.

As an application of Theorem 2, a simple inductive proof yields that any tree is balanceable. Also it is easy to see that the complete graph on 4 vertices is balanceable.

3 Complete graphs

In this section, which is based on results given in [2], we study whether the complete graph K_m is balanceable. We are now considering only graphs with an even number of edges, that is, we will assume that $m \equiv 0, 1 \pmod 4$. It turns out that only K_4 is balanceable.

Observe that a 2-coloring $f : E(K_n) \to \{red, blue\}$ has no balanced K_4 if and only if the red graph R_f (or equivalently the blue graph B_f) is $\{K_{1,3}, K_3 \cup K_1, P_3\}$-free (in the induced sense).

The following lemma characterizes the K_3-free subclass of the family of $\{K_{1,3}, K_3 \cup K_1, P_3\}$-free graphs, and is used in [2] to determine $\mathrm{bal}(n, K_4)$ and $\mathrm{Bal}(n, K_4)$. We define

$$h(n) = \begin{cases} n, & \text{if } n \equiv 0 \pmod 4, \text{ and} \\ n-1, & \text{otherwise.} \end{cases}$$

Lemma 1. *Let G be a $\{K_{1,3}, K_3, P_3\}$-free graph on n vertices. Then each component of G is isomorphic to one of C_4, K_1, K_2 or P_2. Moreover, $e(G) \leq h(n)$, and equality holds if and only if $G \cong J \cup \bigcup_{i=1}^{q} C_4$, where $J \in \{\emptyset, K_1, K_2, P_2\}$ and $q = \lfloor \frac{n}{4} \rfloor$.*

The following lemma shows that the absence of a balanced K_4 in a red-blue coloring of K_n implies that at least one of both, the red graph or the blue graph, is triangle-free.

Lemma 2. *Let $n \geq 5$ and $f : E(K_n) \to \{red, blue\}$ be a coloring without a balanced K_4. Then at least one of R_f or B_f is triangle-free.*

Let $f : E(K_n) \to \{red, blue\}$ be such that $\min\{e(R_f), e(B_f)\} \geq h(n) + 1$ and suppose for contradiction that it has no balanced K_4. Then both R_f and B_f are $\{K_{1,3}, K_3 \cup K_1, P_3\}$-free graphs. By Lemma 2, R_f or B_f is K_3-free. So we may assume, without loss of generality, that R_f is triangle-free. It follows by Lemma 1 that $e(R) \leq h(n)$, which is a contradiction. Hence, $\mathrm{bal}(n, K_4) = h(n)$. This, together with the characterization of the extremal family $\mathrm{Bal}(n, K_4)$, which follows also quite easily by means of the previous lemmas and Mantel's theorem, is given in the following theorem.

Theorem 3. *Let $n \geq 5$. Then $\mathrm{bal}(n, K_4) = h(n)$ and*

$$\mathrm{Bal}(n, K_4) = \left\{ J \cup \bigcup_{i=1}^{\lfloor n/4 \rfloor} C_4 \ : \ J \in \{\emptyset, K_1, K_2, P_2\} \right\}.$$

Theorem 4. *For any positive integer $m \geq 5$ with $m \equiv 0, 1 (\mathrm{mod}\ 4)$, K_m is not balanceable.*

So Theorem 4 solves the question concerning the balanceablility of complete graphs K_m, $m \geq 5$, with an even number of edges.

4 Paths, stars and trees

We know, as shown in Section 2, that all trees are balanceable. In this section, compiling results from [3], we give the precise value of $\mathrm{bal}(n, G)$ and describe $\mathrm{Bal}(n, G)$ for $G = K_{1,k}$ and for $G = P_k$, where $k \geq 2$ is even and n sufficiently large. Moreover, we give a sharp lower bound for $\mathrm{bal}(n, T)$ in the case that T is a tree with k edges, where k is even.

For non-negative integers p and q, let $S_{p,q}$ be the *complete (p, q)-split graph*, that is as a graph whose vertex set is partitioned into two parts, V_1 and V_2, such that $|V_1| = p$, $|V_2| = q$, V_1 induces a complete graph, V_2 is an independent set and such that it contains all edges between the vertices of V_1 and V_2.

The next two theorems give $\mathrm{bal}(n, G)$ and $\mathrm{Bal}(n, G)$ for the case that $G = K_{1,k}$ and $G = P_k$.

Theorem 5. *Let $k \geq 2$ and n be integers with k even and such that $n \geq \max\{3, \frac{k^2}{4} + 1\}$. Then*

$$\mathrm{bal}(n, K_{1,k}) = n\left(\frac{k}{2} - 1\right) - \frac{k^2}{8} + \frac{k}{4} \quad \textit{and} \quad \mathrm{Bal}(n, K_{1,k}) = \left\{ S_{\frac{k}{2}-1, n-\frac{k}{2}+1} \right\}.$$

Theorem 6. *Let $k \geq 2$ and n be integers with k even and such that $n \geq \frac{9}{32}k^2 + \frac{1}{4}k + 1$. Then*

$$\mathrm{bal}(n, P_k) = \begin{cases} \left(\frac{k-2}{4}\right) n - \frac{k^2}{32} + \frac{1}{8}, & \text{for } k \equiv 2 \pmod 4, \\ \left(\frac{k-4}{4}\right) n - \frac{k^2}{32} + \frac{k}{8} + 1, & \text{for } k \equiv 0 \pmod 4, \end{cases}$$

$$\text{and } \mathrm{Bal}(n, P_k) = \left\{ S_{\lfloor \frac{k-2}{4} \rfloor, n - \lfloor \frac{k-2}{4} \rfloor} \right\}.$$

We finish presenting the following sharp upper bound on $\mathrm{bal}(n, T)$, where T is a tree.

Theorem 7. *Let T be a tree on k edges. Then*

$$\mathrm{bal}(n, T) \leq \binom{k/2}{2},$$

and the bound is sharp.

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