On the identifying code number of block graphs

Annegret K. Wagler — University Clermont Auvergne (LIMOS, UMR 6158 CNRS), Clermont-Ferrand, France
G.R. Argiroffo, S.M. Bianchi, Y. Lucarini — Universidad Nacional de Rosario (Dept. de Matemática), Rosario, Argentina

Abstract

The identifying code problem is a search problem that is challenging both from a theoretical and a computational point of view, even for several graphs where other in general hard problems are easy to solve, like bipartite graphs or chordal graphs. Hence, a typical line of attack for this problem is to determine minimum identifying codes of special graphs. In this work we study the problem of determining the cardinality of a minimum identifying code in block graphs (that are diamond-free chordal graphs). We present a linear time algorithm for this problem, as a generalization of a linear time algorithm proposed by Auger in 2010 for the case of trees. Thereby, we provide a subclass of chordal graphs for which the identifying code problem can be solved in linear time. In addition, we discuss an upper bound for the identifying code number of block graphs, state a corresponding conjecture and verify it for some cases.

1 Introduction

Many search problems as, e.g., fault detection in networks or fire detection in buildings, can be modeled by so-called identifying codes in graphs [12].

Consider a graph $G = (V, E)$ and denote by $N[i] = \{i\} \cup N(i)$ the closed neighborhood of a vertex i. A subset $C \subseteq V$ is *dominating* (resp. *separating*) if $N[i] \cap C$ are non-empty (resp. distinct) sets for all $i \in V$. An *identifying code* of G is a vertex subset which is dominating and separating.

Not every graph G admits an identifying code or is *identifiable*: this holds if and only if there are no true twins in G, i.e., there is no pair of distinct vertices $i, j \in V$ with $N[i] = N[j]$ [12]. On the other hand, for every identifiable graph, its whole vertex set trivially forms an identifying code.

The *identifying code number* $\gamma_{ID}(G)$ of a graph G is the minimum cardinality of an identifying code of G. Determining $\gamma_{ID}(G)$ is in general NP-hard [7] and even remains hard for several graph classes where other in general hard problems are easy to solve, including bipartite graphs [7] and two classes of chordal graphs, namely split graphs and interval graphs [9].

The identifying code problem has been actively studied during the last decade, where typical lines of attack are to determine minimum identifying codes of special graphs or to provide bounds for their size. Closed formulas for the exact value of $\gamma_{ID}(G)$ have been found so far only for restricted graph families (e.g. for paths and cycles [5], for stars [10], for complete multipartite graphs [1] and some subclasses of split graphs [3]).

A linear time algorithm to determine $\gamma_{ID}(G)$ if G is a tree was provided by Auger [4]. In this paper, we determine the identifying code number of block graphs (that are diamond-free chordal graphs [6]). We present a linear time algorithm for this problem, as a generalization of the linear time algorithm by Auger for trees. Thereby, we provide a subclass of chordal graphs for which the identifying code problem can be solved in linear time. In addition, we discuss an upper bound for the identifying code number of block graphs, state a corresponding conjecture and verify it for some cases.

2 On the identifying code number of block graphs

A *block graph* is a graph in which every inclusion-wise maximal 2-connected subgraph (block) is a clique (see Figure 1). Block graphs are precisely those chordal graphs in which every two inclusion-wise maximal cliques have at most one vertex in common [11]. Note that a block graph B is identifiable (i.e. has no true twins) if and only if each maximal clique K of B satisfies that all vertices in K, except at most one, have a neighbour that is not in K.

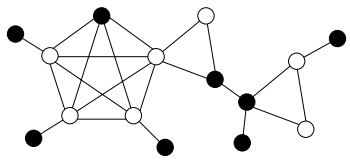


Figure 1: A block graph B (the black vertices form an identifying code of B).

In order to provide a linear algorithm which computes $\gamma_{ID}(B)$ of any block graph B, we adopt the following notation from [4]. Let $G = (V, E)$ be a graph and $v \in V$, then $C \subseteq V$ is a $\{v\}$-*almost identifying code of* G if the sets $C \cap N[u]$ are nonempty and pairwise distinct for all $u \in V - \{v\}$. As in [4], we consider several functions related to identification and domination properties of $\{v\}$-almost identifying codes.

The idea of our main algorithm ICB is as follows: given a connected block graph B, the algorithm randomly selects a vertex v_1 from B and calls a subroutine RICB(v_1, B) that computes the values of the studied functions in a recursive manner in smaller and smaller block graphs.

The subroutine RICB chooses a maximal clique K containing v_1 and either returns the initial function values (if $K = \{v_1\}$) or deletes all edges of $K = \{v_1, \ldots, v_k\}$ and calls recursively RICB(v_i, B_i) for all so-obtained components B_i of $B - K$. In a recomposition step, the function values for B are computed from the function values returned from RICB(v_i, B_i) for all components. We can show:

Theorem 1. *[2] Algorithm ICB computes $\gamma_{ID}(B)$ of an identifiable block graph B in linear time (or returns ∞ if no identifying code exists in B).*

Note that ICB is a generalization of the algorithm proposed by Auger [4]. It takes into account the identifiable condition for block graphs that is not needed in the case of trees such that the recomposition step of our algorithm is more involved and uses different functions as in [4].

Moreover, we are intersted in an upper bound for the identifying code number of block graphs. Let $n_Q(G)$ denote the number of maximal cliques of a graph G. We conjecture:

Conjecture 1. *For any identifiable block graph B, we have that $\gamma_{ID}(B) \leq n_Q(B)$.*

There are some families of block graphs where this bound is known to be true.

First of all, recall that every tree is a block graph (with clique size 2). We clearly have $n_Q(T) = |V| - 1$ for any tree $T = (V, E)$. It is known from [8] that $\gamma^{ID}(G) \leq |V| - 1$ holds for any connected graph $G = (V, E)$ (with stars being examples where this bound is tight). This immediately implies for any tree T that

$$\gamma_{ID}(T) \leq |V| - 1 = n_Q(T).$$

A *thin headless spider* is a graph G whose node set can be partitioned into S and Q, where $S = \{s_1, \ldots, s_k\}$ is a stable set, $Q = \{q_1, \ldots, q_k\}$ is a clique, and s_i is adjacent to q_j if and only if $i = j$. Every thin headless spider is both a split graph and a block graph.

We clearly have $n_Q(G) = |S| + 1$. In [1, 3] it was shown that for every identifying code C of such a graph, we have $|C \cap Q| \geq 2$ and $|C \cap S| \geq |S| - 1$, and that there is always a code C with $|C| = |S| + 1$. Together, this implies that for every thin headless spider G, it holds that

$$\gamma_{ID}(G) = |S| + 1 = n_Q(G).$$

Moreover, we call a graph $G = (V, E)$ a *double sun* if the vertex set can be partitioned into $V = U \cup U' \cup V'$ with $|U| = |U'| = |V'| = k$, each $v_i \in V'$ has exactly one neighbor $u_i \in U$ and exactly one neighbor $u'_i \in U'$, and $U \cup U'$ forms a stable set (whereas there is no restriction on the subgraph induced by V'). We can show:

Lemma 1. *For any double sun $G = (U \cup U' \cup V', E)$, the vertex subset $U \cup U'$ forms a minimum identifying code.*

We call a double sun $G = (U \cup U' \cup V', E)$ a *double clique-sun* if V' induces a clique. Clearly, double clique-suns G are both split graphs and block graphs and we have

$$\gamma_{ID}(G) = |U \cup U'| = 2k < 2k + 1 = n_Q(G).$$

Furthermore, recall that a block graph B is identifiable if and only if every maximal clique K of B has at most one vertex v_K without neighbors outside K. We call a block graph *critical identifiable* if every maximal clique K has exactly one such vertex v_K without neighbors outside K. Note that stars $K_{1,k}$ are the only trees that satisfy this condition. We can show:

Lemma 2. *In a critical identifiable block graph B, the set of all such vertices v_K is an identifying code of B, hence $\gamma_{ID}(B) \leq n_Q(B)$ holds.*

Note that the studied bound is tight for stars and thin headless spiders, but neither for paths, nor double clique-suns, nor critical identifiable block graphs, see Figure 2 for examples.

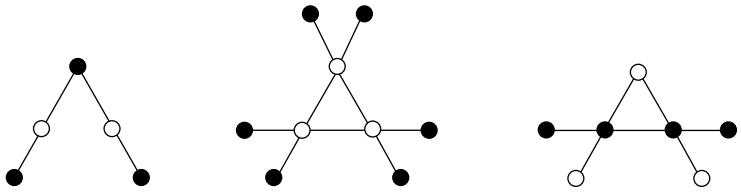


Figure 2: Block graphs B with $\gamma_{ID}(B) < n_Q(B)$ (the black vertices form an identifying code of B): a path, a double clique-sun, a critical identifiable block graph.

We further note that the gap between $\gamma_{ID}(B)$ and $n_Q(B)$ can be larger than one. This clearly applies to longer paths (where $\gamma_{ID}(P_n) = \lceil \frac{n+1}{2} \rceil$ by [5] and $n_Q(P_n) = n - 1$), but also for other block graphs (e.g. obtained from those shown in Figure 2 by identification in one vertex).

3 Concluding remarks

The identifying code problem is a challenging problem and even remains hard for several graph classes where other in general hard problems are easy to solve, including bipartite graphs [7] and two classes of chordal graphs, split graphs and interval graphs due to [9].

In this paper, we determine the identifying code number of block graphs (that are diamond-free chordal graphs). We present a linear time algorithm for this problem, as a generalization of the linear time algorithm proposed by Auger [4] for trees. Thus, we provide a subclass of chordal graphs for which the identifying code problem can be solved in linear time. Moreover, we conjecture that the number of maximal cliques is always an upper bound for the identifying code number of any block graph. We verify this conjecture for some cases and present examples where it is tight resp. not tight. Our lines of future research include to verify/falsify this conjecture.

Furthermore, note that our algorithm ICB only returns the identifying code number of block graphs, but not an identifying code of minimum size. Our lines of future research, thus, include to adapt our algorithm accordingly.

Finally, it is interesting whether similar ideas work for graph classes with a similar structure, e.g. for cacti (graphs in which every maximal 2-connected subgraph is an edge or a cycle) or for block-cacti (graphs in which every maximal 2-connected subgraph is a clique or a cycle).

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