On essentially 4-edge-connected cubic bricks

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Abstract

Lovász (1987) proved that every matching covered graph may be uniquely decomposed into a list of bricks (nonbipartite) and braces (bipartite). Carvalho, Lucchesi and Murty (2002) proved a conjecture of Lovász which states that every brick G, distinct from K_4, the triangular prism $\overline{C_6}$ and the Petersen graph, has a b-invariant edge e — that is, an edge e such that $G-e$ is a matching covered graph with exactly one brick.

A cubic graph is essentially 4-edge-connected if it is 3-edge-connected and if the only 3-cuts are the trivial ones. A brick G is near-bipartite if it has a pair of edges $\{e, f\}$ such that $G-e-f$ is bipartite matching covered; for instance, K_4 and $\overline{C_6}$. We prove that every essentially 4-edge-connected cubic graph G is either a brick or a brace; furthermore, if G is a brick that is not near-bipartite and is not the Petersen graph, then G has at least $\frac{|V(G)|}{2}$ b-invariant edges.

1 Motivation

A connected graph, on at least two vertices, is *matching covered* if each edge lies in some perfect matching. In his seminal paper characterizing the matching lattice, Lovász [1] proved that every matching covered graph may be "decomposed" uniquely into a list of special matching covered graphs called *bricks* (nonbipartite) and *braces* (bipartite). This "decomposition" is known as the *tight cut decomposition*.

When investigating problems in matching theory, often one may restrict attention to matching covered graphs; furthermore, in several cases, it suffices to solve the problem for bricks and for braces. A classical example is the problem of characterizing, or of recognizing, Pfaffian graphs. A matching covered graph is Pfaffian if and only if each of its bricks and braces is Pfaffian; see [2, 3].

An edge e of a brick G is *b-invariant* if $G-e$ is a matching covered graph whose tight cut decomposition results in precisely one brick. Confirming a conjecture of Lovász, the following was proved by Carvalho, Lucchesi and Murty [6].

Theorem 1. *Every brick, distinct from K_4, the triangular prism $\overline{C_6}$ and the Petersen graph, has a b-invariant edge.*

As an application of the above theorem, Carvalho, Lucchesi and Murty [7] gave an alternative proof for the characterization of the matching lattice, and proved other deep results concerning *ear decompositions* of matching covered graphs. Since then, the existence of b-invariant edges in bricks, as well as the existence of special types of b-invariant edges (such as *thin* and *strictly thin* edges), have found many applications in matching theory; see [9, 11, 12].

In certain applications, it is helpful to have the presence of "many" b-invariant edges — for instance, one incident with each vertex. A graph is *solid* if its perfect matching polytope is described by non-negativity and degree constraints; see [8]. The notion of solid graphs is a generalization of bipartite graphs that has played an important role in many works of Carvalho, Lucchesi and Murty[9, 11]. An immediate consequence of Theorem 3.3 and Corollary 6.12 in [11] is the following.

Theorem 2. *In a solid brick G, distinct from K_4, each vertex is incident with at least one b-invariant edge; consequently, G has at least $\frac{|V(G)|}{2}$ b-invariant edges.*

In this paper, we prove similar results for another rich class of bricks, namely the class of *essentially 4-edge-connected cubic bricks*. See Theorem 5 and Corollary 1. This paper is part of an on-going project wherein we hope to characterize the Pfaffian cubic graphs.

2 Tight Cut Decomposition

For a nonempty proper subset X of the vertices of a graph G, we denote by $\partial(X)$ the cut associated with X. We refer to X and $\overline{X} := V(G) - X$ as the *shores* of $\partial(X)$. A cut is *trivial* if any of its shores is a singleton. We say that $\partial(X)$ is a *k-cut* if $|\partial(X)| = k$. For a cut $\partial(X)$, we use G/X to denote the graph obtained by contracting the shore X to a single vertex; the graph $G/\overline{X}$ is defined analogously. The two graphs G/X and $G/\overline{X}$ are called the *$\partial(X)$-contractions* of G.

Let G be a matching covered graph. A cut $\partial(X)$ is *tight* if $|M \cap \partial(X)| = 1$ for every perfect matching M of G. It is easily verified that if $\partial(X)$ is a nontrivial tight cut of G, then each $\partial(X)$-contraction is a matching covered graph that has strictly fewer vertices than G. We can repeat this procedure recursively until we obtain a list of matching covered graphs — each of which is free of nontrivial tight cuts — these are called *bricks* (nonbipartite), and *braces* (bipartite). This procedure is known as a *tight cut decomposition* of G. In Figure 1 (left), the three edges crossing the bold line constitute a nontrivial tight cut, say $\partial(X)$; the two $\partial(X)$-contractions are K_4 and $K_{3,3}$.

In general, a matching covered graph may admit several tight cut decompositions. However, Lovász [1] proved the remarkable result that any two tight cut decompositions of a matching covered graph yield the same list of bricks and braces (except possibly for multiplicities of edges). In particular, any two tight cut decompositions of a matching covered graph G yield the same number of bricks; this number is denoted by $b(G)$. Note that $b(G) = 0$ if and only if G is bipartite.

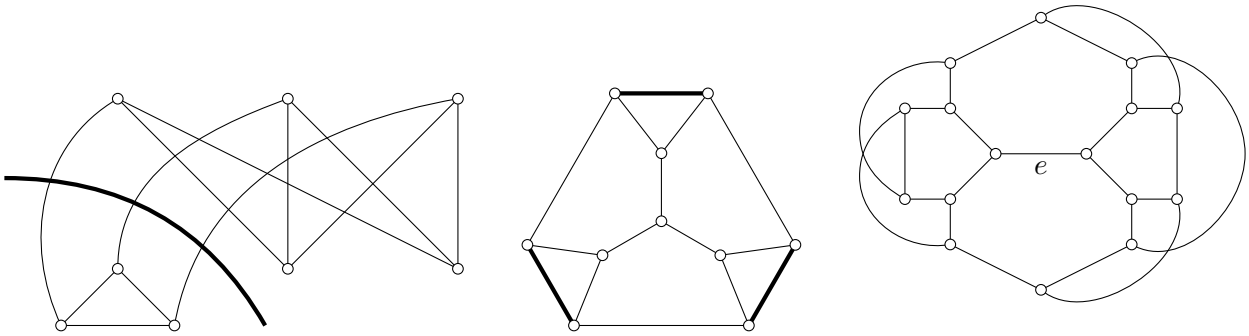


Figure 1: (left) A nontrivial tight cut; (center) The Tricorn has exactly three removable edges and no removable doubletons; (right) The removable edge e is neither b-invariant nor quasi b-invariant

It is easily shown, using Tutte's 1-factor Theorem, that every 2-edge-connected cubic graph is matching covered. However, in general, such a graph need not be a brick or a brace; see Figure 1 (left). We say that a graph is *essentially 4-edge-connected* if it is 3-edge-connected and if it is free of nontrivial 3-cuts. Our first result is the following.

Theorem 3. *Every essentially 4-edge-connected cubic graph is either a brick or a brace.*

The proof is straightforward, and was known to C. N. Campos and C. L. Lucchesi in 1999. Braces are bipartite, and are relatively well-understood. In the rest of this paper, we investigate essentially 4-edge-connected cubic bricks — a few popular examples are nonbipartite Möbius ladders (including K_4), nonbipartite prisms (except $\overline{C_6}$), and the Petersen graph; see [12] for definitions.

3 Removable classes

Let G be a brick. An edge e is *removable* if $G - e$ is matching covered; it is easily shown that $G - e$ is nonbipartite; in particular, $b(G - e) \geq 1$. A pair of edges $\{e, f\}$ is a *removable doubleton* if $G - e - f$ is bipartite and matching covered; clearly, in this case, neither e nor f is removable.

A brick G is *near-bipartite* if it has a removable doubleton; otherwise G is *non-near-bipartite*. For instance, the Petersen graph is non-near-bipartite. On the other hand, each of K_4 and $\overline{C_6}$ has three distinct removable doubletons; furthermore, each of them is devoid of removable edges. Lovász [1] proved that every brick distinct from K_4 and $\overline{C_6}$ has a removable edge.

A *removable class* of a brick is either a removable edge or a removable doubleton — each of which is a specialization of the more general notion of a *removable ear* or *double ear* in matching covered graphs. For an in-depth treatment of this topic, we refer the reader to Carvalho, Lucchesi and Murty [4] wherein they prove that each brick G has at least $\Delta(G)$ removable classes, and at least $\Delta(G) - 2$ removable edges. (Here, $\Delta(G)$ denotes the maximum degree of G.)

For a removable edge e of a brick G, the quantity $b(G - e)$ may be arbitrarily large. Recall that e is *b-invariant* if $b(G - e) = 1$. We say that e is *quasi-b-invariant* if $b(G - e) = 2$. It is easily verified that each edge of the Petersen graph is quasi-b-invariant. Our next result is the following.

Theorem 4. *Let G be an essentially 4-edge-connected cubic brick and let e be any edge of G. Then exactly one of the following holds:*

(i) either e participates in a removable doubleton,

(ii) or e is b-invariant,

(iii) or otherwise e is quasi-b-invariant.

It is worth noting that the conclusion of Theorem 4 does not hold for all cubic bricks; essentially 4-edge-connectedness is crucial. A cubic brick G may have an edge e that is neither removable nor participates in a removable doubleton; see Figure 1(center). Furthermore, a cubic brick G may have a removable edge e such that $b(G - e) \geq 3$; see Figure 1(right).

As mentioned in Section 1, it is often helpful to have the presence of "many" b-invariant edges. On a related note, a brick being near-bipartite is most likely "good" news. For example, while there has been no significant progress in characterizing Pfaffian bricks; Fischer and Little [5] were able to characterize Pfaffian near-bipartite bricks. In this sense, the only "unpleasant" outcome for any particular edge e of an essentially 4-edge-connected cubic brick G is outcome *(iii)* as stated in Theorem 4. With this in mind, we started wondering whether we could prove an upper bound on the number of quasi-b-invariant edges in an essentially 4-edge-connected cubic brick.

The Petersen graph has the property that if v is any vertex then v is incident with three edges — each of which is quasi-b-invariant. We were able to prove that, among the essentially 4-edge-connected cubic bricks, the Petersen graph is unique with respect to this property. In fact, we prove a stronger result.

Theorem 5 (Main Theorem)**.** *Let G be an essentially 4-edge-connected cubic brick, let v be any vertex of G and let e_1, e_2, e_3 be the three distinct edges incident with v. If e_1 and e_2 are both quasi-b-invariant then exactly one of the following holds:*

(i) either e_3 participates in a removable doubleton and G is the Cubeplex,

(ii) or e_3 is quasi-b-invariant and G is the Petersen graph,

(iii) or otherwise e_3 is b-invariant.

The Cubeplex is a graph on 12 vertices that first appeared in the works of Fischer and Little [5] where they showed that it is one of two *minimally non-Pfaffian* near-bipartite bricks; they referred to the two graphs as Γ_1 and Γ_2. The name Cubeplex for Γ_1 is due to Norine and Thomas [10]. The following is an immediate consequence of Theorem 5.

Corollary 1. *In an essentially 4-edge-connected cubic non-near-bipartite brick G, distinct from the Petersen graph, each vertex is incident with at least one b-invariant edge; consequently, G has at least $\frac{|V(G)|}{2}$ b-invariant edges.*

We conjecture that the lower bound of $\frac{|V(G)|}{2}$ holds for all essentially 4-edge-connected cubic bricks, even those that are near-bipartite except K_4.

The proofs of Theorems 3, 4 and 5 are available here [13].

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