Cycles Containing All the Odd-Degree Vertices

Kathie Cameron — **Wilfrid Laurier University**

Abstract

I prove that if G is a finite simple graph in which no two even-degree vertices are adjacent, and if e is an edge of G, then the number of cycles containing e and all the odd-degree vertices is even. Andrew Thomason proved this when there are no even-degree vertices using his elegant lollipop method. Thomason's Theorem itself generalizes Smith's Theorem, which is the instance in which all vertices have degree 3.

This work was inspired by a recent result of Carsten Thomassen, who proved that in a graph in which no two even-degree vertices are adjacent, if there is one cycle containing all the odd-degree vertices, then there is another.

1 Introduction

All graphs in this paper are finite and have no loops or multiple edges. A *hamiltonian path* in a graph G is a path which contains each vertex (exactly once); a *hamiltonian cycle* in G consists of a hamiltonian path together with an edge joining the last vertex of the path to the first.

In 1946, Bill Tutte [7] gave a beautiful short proof of Smith's Theorem:

Theorem 1. *Smith's Theorem. Let G be a 3-regular graph and let e be an edge of G. The number of hamiltonian cycles of G containing e is even.*

An obvious corollary of any theorem which says that the number of objects is even is: Given one of the objects, there exists another. Tutte's proof does not provide an algorithm for given one hamiltonian cycle containing e, finding another.

In 1978, Andrew Thomason [5] extended Smith's Theorem to any graph where all vertices have odd degree:

Theorem 2. (Andrew Thomason, [5]) *Let G be a graph where all vertices have odd degree and let e be an edge of G. The number of hamiltonian cycles of G containing e is even.*

Andrew Thomason's proof constructs a graph $X(G)$ which he calls a lollipop graph such that the odd-degree vertices of $X(G)$ correspond precisely to the hamiltonian cycles of G containing e. This provides an algorithm for finding a second hamiltonian cycle containing e by walking in $X(G)$ from a given odd-degree vertex to another odd-degree vertex. Unfortunately, this elegant algorithm is exponential [1, 4], even for 3-regular graphs. Jack Edmonds and I use the term *exchange graph* for a graph like Thomason's in which the odd-degree vertices correspond to the objects of interest. Many theorems which say that the number of objects is even can be proved by constructing an exchange graph [2].

Recently, Carsten Thomassen [6] proved:

Theorem 3. (Carsten Thomassen, [6]) *Let G be a graph where no two even-degree vertices are adjacent. If there is one cycle C containing all the odd-degree vertices of G, then there is another such cycle.*

In fact, Thomassen's proof actually shows:

Theorem 4. *Let G be a graph where no two even-degree vertices are adjacent and let e be an edge of G. If there is one cycle C containing e and all the odd-degree vertices of G, then there is another such cycle.*

Knowing Thomassen's Theorem, it is natural to ask: Is the number of cycles containing e and all the odd-degree vertices even? The answer is yes, and is a corollary of the main result in this paper:

Theorem 5. *Let G be a bipartite graph with bipartition (Y, Z) where every vertex in Y has even degree and every vertex in Z has odd degree. Let e be an edge of G. Then the number of cycles containing e as well as all the odd-degree vertices is even.*

2 Exchange Operations

I'll say a path $v_1, v_2, \ldots, v_k$ in G is *extendible to a cycle* if $v_k v_1$ is an edge of G.

To prove Theorem 2, Andrew Thomason constructs a graph $X(G)$ whose vertices are the hamiltonian paths in G beginning with the edge $e = v_1v_2$ (in order). Two vertices (hamiltonian paths) are adjacent in $X(G)$ if one can be obtained from the other by the following exchange operation: Given a hamiltonian path P with vertices $v_1, v_2, \ldots v_n = z$ (in order), add an edge zv_i where $i \neq 1$ and remove the edge v_iv_{i+1} to get a new hamiltonian path P' beginning with v_1v_2. (Note that this operation is reversible.) It can easily be seen that if P is extendible to a hamiltonian cycle in G, then the degree of the vertex corresponding to P in $X(G)$ is $\deg_G(z) - 2$, which is odd, and if P is not extendible to a hamiltonian cycle in G, then the degree of the vertex corresponding to P in $X(G)$ is $\deg_G(z) - 1$, which is even. Since in any graph, the number of vertices of odd degree is even, it follows that the number of hamiltonian paths beginning with v_1 followed by v_2 which are extendible to a hamiltonian cycle is even, and that is the number of hamiltonian cycles containing $e = v_1v_2$.

Regardless of whether path P is hamiltonian, I will call such an exchange an A-exchange. More precisely, an A-exchange is: given a path P with vertices $v_1, v_2, \ldots v_k = z$, add an edge zv_i where $i \neq 1$ and v_i is on P and remove the edge v_iv_{i+1} to get a new path P' containing the same vertices as P.

Carsten Thomassen's proof of Theorem 3 used A-exchanges as well as three other exchanges, one of which I will call a C-exchange, and define as follows. Given a path P with vertices $v_1, v_2, \ldots v_k = z$, add a path z, r, v_i where r is a vertex not in P and v_i is a vertex of P different from v_1; remove vertex v_{i+1} and edges v_iv_{i+1} and $v_{i+1}v_{i+2}$ from P to obtain a new path P''. (Note that C-exchanges are reversible: from path P'' ending at v_{i+2}, add path v_{i+2}, v_{i+1}, v_i and remove vertex r and edges rv_i and rz to obtain P.)

3 Cycles Containing all the Odd-Degree Vertices

I will now prove Theorem 5.

Proof. As in the statement of the theorem, let G be a bipartite graph with bipartition (Y, Z) where every vertex in Y has even degree and every vertex in Z has odd degree. Let $e = v_1v_2$ be an edge of G and let v_1 be the even-degree end of e.

Construct an exchange graph $X(G)$ as follows: The vertices of G are paths beginning with v_1 followed by v_2, containing all odd-degree vertices of G and ending at an odd-degree vertex. Two

such paths are adjacent in G if one can be obtained from another by either an A-exchange or a C-exchange.

Consider a path P beginning with v_1 followed by v_2, containing all odd-degree vertices of G and ending at an odd-degree vertex z. Consider an edge meeting z whose other end t is on P and is not v_1. Call such edges type 1. One exchange (an A-exchange) can be made with zt.

Now consider an edge meeting z whose other end r is not in P. Call such edges type 2. Then r has even degree, and where S is the set of vertices adjacent to r distinct from z, $|S|$ is odd, each vertex of S has odd-degree, and v_1 is not in S (since v_1 has even degree). For each edge rt where t is in S one exchange (a C-exchange) can be made, so an odd number of exchanges can be made using zr. Because the degree of z is odd,

> if P is a path such that zv_1 is an edge (i.e. if P is extendible to a cycle), then the number of edges meeting z which can be used in an exchange is odd (all except zv_1 and the edge of P meeting z).
>
> > If the number of type 1 edges is odd, then the number of type 2 edges is even, so the number of exchanges that can be made from z is odd + (even $\times$ odd) which is odd.
> >
> > If the number of type 1 edges is even, then the number of type 2 edges is odd, so the number of exchanges that can be made from z is even + (odd $\times$ odd) which is odd.
>
> If P is a path such that zv_1 is not an edge (i.e. if P is not extendible to a cycle), then the number of edges meeting z which can be used in an exchange is even (all except the edge of P meeting z).
>
> > If the number of type 1 edges is odd, then the number of type 2 edges is odd, so the number of exchanges that can be made from z is odd + (odd $\times$ odd) which is even.
> >
> > If the number of type 1 edges is even, then the number of type 2 edges is even, so the number of exchanges that can be made from z is even + (even $\times$ odd) which is even.

Thus the degree of vertices of $X(G)$ corresponding to paths in G which are extendible to cycles is odd and the degree of vertices of $X(G)$ corresponding to paths in G which are not extendible to cycles is even. Since in any graph the number of vertices of odd degree is even, it follows that the number of cycles of G containing $e = v_1v_2$ and all odd-degree vertices is even. □

Corollary 1. *Suppose H is a graph in which no two even-degree vertices are adjacent and let e be an edge of H. Then the number of cycles containing e and all the odd-degree vertices of H is even.*

Proof. Let H be a graph in which no two even-degree vertices are adjacent. Split each edge uv joining odd-degree vertices as follows: Replace edge uv by a path of two edges consisting of a new vertex w and two edges uw and wv; call the set of new vertices W. This results in a bipartite graph G with bipartition (Y, Z) where Z is the set of odd-degree vertices of H and Y consists of the even-degree vertices of H together with W. A cycle of H corresponds precisely to a cycle of G by contracting to an edge each path of length 2 whose midpoint is a vertex of W. In particular, if uv is an edge of H joining odd-degree vertices, and a cycle of G contains an edge uw where w is in W, then it must contain wv. Thus Corollary 1 follows from Theorem 5. □

Jack Edmonds and I have generalized Theorem 5 to one concerning trees with specified degrees in a bipartite graph [3].

References

[1] Kathie Cameron. Thomason's algorithm for finding a second hamiltonian circuit through a given edge in a cubic graph is exponential on Krawczyk's graphs. Discrete Mathematics 235(1-3) (2001), 69-77.

[2] Kathie Cameron and Jack Edmonds. Some graphic uses of an even number of odd nodes. Annales de l'Institut Fourier 49(3) (1999), 815-827.

[3] Kathie Cameron and Jack Edmonds. A PPA episode of exchange graphs. International Symposium on Combinatorial Optimization (ISCO 2018).

[4] Adam Krawczyk. The complexity of finding a second Hamiltonian cycle in cubic graphs. J. Comput. System Sci. 58(3) (1999), 641-647.

[5] A. G. Thomason. Hamiltonian cycles and uniquely edge colourable graphs. Ann. Discrete Math. 3 (1978), 259-268.

[6] Carsten Thomassen. Chords in longest cycles. Journal of Combinatorial Theory, Series B 129 (2018), 148-157.

[7] W. T. Tutte. On Hamiltonian circuits. J. London Math. Soc. 21 (1946), 98-101.