# Rainbow vertex coloring bipartite graphs and chordal graphs

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**Abstract**

Given a vertex-colored graph, a path is called a rainbow vertex path if all its internal vertices have distinct colors. A graph $G$ is (strongly) rainbow vertex-connected if there is a rainbow vertex (shortest) path between every pair of its vertices. We study the problem of deciding whether the vertices of a given graph $G$ can be colored with at most $k$ colors so that $G$ is (strongly) rainbow vertex-connected. Although edge colorings have been studied extensively under similar constraints, there are significantly fewer results on the vertex variants that we consider. We show that the problems are NP-complete and hard to approximate on bipartite graphs and on chordal graphs, particularly even on bipartite apex graphs and on split graphs. On the positive side, we give polynomial-time algorithms on bipartite permutation graphs, interval graphs, and block graphs.

## 1 Introduction

Many different types of graph colorings and connectivity measures have been studied throughout time. The concept of rainbow coloring brings these two extensively studied core topics together, and it was first defined a decade ago by Chartrand et al. [2] on edge-colored graphs. In 2010, a variant of rainbow coloring was defined by Krivelevich and Yuster [6] on *vertex-colored* graphs. Let $G$ be a connected, vertex-colored graph. A *rainbow vertex path* is a path such that all its *internal* vertices have distinct colors. We say that $G$ is *rainbow vertex-connected* if there is a rainbow vertex path between every pair of vertices of $G$. The resulting computational problem Rainbow Vertex Coloring (RVC) is the decision problem in which we are given a connected graph $G$ and an integer $k$, and the task is to decide whether the vertices of $G$ can be colored with at most $k$ colors such that $G$ is rainbow vertex-connected. The concept of rainbow coloring has various applications in telecommunications, data transfer, and encryption (see [7] and the references therein).

RVC is known to be NP-complete for every $k \geq 2$ [4, 3]. Moreover, Eiben et al. [5] showed that there is no polynomial-time approximation algorithm for the problem with a factor less than 2 unless $\mathsf{P} \neq \mathsf{NP}$. A natural variant of the problem concerning the existence of rainbow vertex shortest paths is defined as follows. A vertex-colored graph $G$ is *strongly rainbow vertex-connected* if between every pair of vertices of $G$, there is a *shortest* path that is rainbow vertex. The Strong Rainbow Vertex Coloring (SRVC) problem takes as input a connected graph $G$ and an integer $k$, and the task is to decide whether the vertices of $G$ can be colored such that $G$ is strongly rainbow vertex-connected. This problem is also NP-complete for every $k \geq 2$ [5]. However, it should be noted that the complexity of RVC and SRVC need not be the same on a given graph class [7].

While the edge variant of the problem has been widely studied with more than 300 published papers, there are considerably fewer results on RVC and SRVC. In particular, very few results are known for the problems on structured graph classes. We contribute in this direction by giving hardness results and polynomial-time algorithms for RVC and SRVC on bipartite graphs and

chordal graphs, and some of their subclasses. We begin by showing that both problems are NP-complete and hard to approximate on bipartite apex graphs. To complement these results, we show that the problems can be solved in polynomial time on bipartite permutation graphs. In addition, we prove that both problems are NP-complete and hard to approximate on split graphs, and consequently also on chordal graphs. On the positive side, we give polynomial-time algorithms for both problems on unit interval graphs and on block graphs, as well as a polynomial-time algorithm for RVC on interval graphs.

## 2 Preliminaries

In this paper, we work on undirected simple graphs. We refer to [1] for basic definitions and terminology concerning graph theory. We write $[n]$ to denote $\{1, 2, \ldots, n\}$ for a positive integer $n$.

A *$k$-coloring* of a graph $G = (V, E)$ is a function $c : V \to [k]$. A *coloring* is a $k$-coloring for some $k \leq n$, while a *proper coloring* has $c(u) \neq c(v)$ for every $uv \in E$. The Graph Coloring problem takes as input a graph $G$ and an integer $k$ and asks whether there is a proper coloring of $G$ with at most $k$ colors. The problem is well-known to be NP-complete for every $k \geq 3$ [9] and also NP-hard to approximate within a factor of $n^{1-\varepsilon}$ for any $\varepsilon > 0$, where $n$ is the number of vertices [10].

A coloring $c$ is a *(strong) rainbow vertex coloring* of $G$ if $G$ is (strongly) rainbow vertex-connected under $c$. The *rainbow vertex connection number* of $G$, denoted by $\mathbf{rvc}(G)$, is the minimum $k$ such that $G$ has a rainbow vertex coloring with $k$ colors. Analogously, the *strong rainbow vertex connection number* of $G$, denoted by $\mathbf{srvc}(G)$, is the minimum $k$ such that $G$ has a strong rainbow vertex coloring with $k$ colors. The parameter $\mathbf{srvc}(G)$ was defined by Li et al. [8], and they also verified that $\mathrm{diam}(G) - 1 \leq \mathbf{rvc}(G) \leq \mathbf{srvc}(G) \leq n - 2$, where $\mathrm{diam}(G)$ denotes the diameter of $G$.

For our hardness reductions we will use the well-known NP-complete problem Hypergraph Coloring. This problem takes as input a hypergraph $H$ and an integer $k$ and asks whether there is a $k$-coloring of the vertices of $H$ such that no hyperedge is monochromatic. The problem is well-known to be NP-complete for every $k \geq 2$ [9].

## 3 Bipartite graphs

In this section, we show that RVC and SRVC are hard on bipartite graphs, for $k \geq 3$. We complement these results by showing that both problems can be solved in linear time on bipartite permutation graphs. We start with a simple observation that deciding whether $\mathbf{rvc}(G) \leq 2$ or $\mathbf{srvc}(G) \leq 2$ is easy on bipartite graphs.

**Proposition 1.** *If $G$ is a bipartite graph with $\mathrm{diam}(G) = 3$, then $\mathbf{rvc}(G) = \mathbf{srvc}(G) = 2$. Moreover, such a coloring can be found in linear time.*

It turns out that if $\mathrm{diam}(G) \geq 4$, then $\mathbf{rvc}(G)$ and $\mathbf{srvc}(G)$ of a bipartite graph $G$ become much harder to compute.

**Theorem 1.** *Let $G$ be a bipartite graph of diameter 4. It is NP-complete to decide both whether $\mathbf{rvc}(G) \leq k$ and whether $\mathbf{srvc}(G) \leq k$, for every $k \geq 3$. Moreover, it is NP-hard to approximate both $\mathbf{rvc}(G)$ and $\mathbf{srvc}(G)$ within a factor of $n^{1/3-\varepsilon}$, for every $\varepsilon > 0$.*

*Sketch of the proof.* We prove the statements by a reduction from Hypergraph Coloring. Let $H = (N, \mathcal{E})$ be an arbitrary hypergraph and let $m = |N|$. We construct a bipartite graph

$G = (V = \{a\} \cup N' \cup I,\ E)$ where $N' = N_1 \cup \cdots \cup N_{m+1}$, $I' = I_1 \cup \cdots \cup I_{m+1}$, $N_i := \{v_i \mid v \in N\}$, $I_i := \{x_e^i \mid e \in \mathcal{E}\}$ and $E := \{av_i \mid v \in N, i \in [m+1]\} \cup \{v_i x_e^i \mid v \in N, e \in \mathcal{E}, i \in [m+1]$ and vertex $v$ belongs to hyperedge $e$ in $H\}$. A bipartition of $G$ is given by $(\{a\} \cup I,\ N')$. Observe that $\mathrm{diam}(G) = 4$.

**Lemma 1.** *For all $2 \leq k \leq n-1$, hypergraph $H$ has a proper $k$-coloring if and only if $\mathbf{rvc}(G) \leq \mathbf{srvc}(G) \leq k+1$.*

The statements of the theorem follow from the above lemma as follows. For membership in NP, a certificate that $\mathbf{rvc}(G) \leq k$ consists of a $k$-coloring and a list of rainbow vertex paths, one for every pair of non-adjacent vertices. Since Hypergraph Coloring is NP-complete for each $k \geq 2$, the first part of the theorem follows from Lemma 1. Now, suppose we restrict the input hypergraph $H$ to simple graphs, i.e., all hyperedges are of size 2, then the reduction is from Graph Coloring. The second statement of the theorem follows from Lemma 1 together with the hardness of approximation of Graph Coloring [10]. The analogous statements for $\mathbf{srvc}(G)$ follow by verifying that the previous arguments work in the case of shortest paths. □

Yet another corollary follows from the proof of Theorem 1, which we would like to highlight in particular, since very little is known in terms of hardness on sparse graph classes for any of the variants of the rainbow problems. The corollary concerns the class of apex graphs. A graph is an *apex graph* if it contains a vertex whose removal results in a planar graph.

**Corollary 1.** *Let $G$ be a bipartite apex graph of diameter 4. It is NP-complete to decide both whether $\mathbf{rvc}(G) \leq 4$ and whether $\mathbf{srvc}(G) \leq 4$.*

*Proof.* Restrict the input hypergraph $H$ in the proof of Theorem 1 to be a planar graph. The graph resulting from the construction given in that proof is a bipartite apex graph. The statement follows from Lemma 1 and the fact that Planar 3-Coloring is NP-complete. □

We finish this section by complementing the above hardness results with a positive result for bipartite permutation graphs. A bipartite graph is a *bipartite permutation graph* if it admits an intersection model consisting of straight lines between two parallel lines.

**Theorem 2.** *If $G$ is a bipartite permutation graph, then $\mathbf{rvc}(G) = \mathbf{srvc}(G) = \mathrm{diam}(G) - 1$, and the corresponding (strong) rainbow vertex coloring can be found in linear time.*

## 4 Chordal graphs

In this section, we study the complexity of RVC and SRVC on chordal graphs and some subclasses of chordal graphs. A graph is *chordal* if all of its induced cycles are of length 3. A graph is a *split graph* if its vertex set can be partitioned into an independent set and a clique. Split graphs form a subclass of chordal graphs. We start by proving that both problems are NP-complete when the input graph is a split graph.

**Theorem 3.** *Let $G$ be a split graph of diameter 3. It is NP-complete to decide both whether $\mathbf{rvc}(G) \leq k$ and whether $\mathbf{srvc}(G) \leq k$, for every $k \geq 2$. Moreover, it is NP-hard to approximate both $\mathbf{rvc}(G)$ and $\mathbf{srvc}(G)$ within a factor of $n^{1/3-\varepsilon}$, for every $\varepsilon > 0$.*

*Sketch of the proof.* Again, we give a reduction from Hypergraph Coloring. Let $H = (N, \mathcal{E})$ be an arbitrary hypergraph and let $m = |N|$. We construct a split graph $G = (V = N' \cup I,\ E)$ where $N' = N_1 \cup \cdots \cup N_{m+1}$, $I' = I_1 \cup \cdots \cup I_{m+1}$, $N_i := \{v_i \mid v \in N\}$, $I_i := \{x_e^i \mid e \in \mathcal{E}\}$ and $E := \{v_i x_e^i \mid v \in N, e \in \mathcal{E}, i \in [m+1]$ and vertex $v$ belongs to hyperedge $e$ in $H\} \cup \{u_i v_j \mid u, v \in N,\ i, j \in [m+1]\}$. The constructed graph $G$ is a split graph since $G[I]$ is an independent set and $G[N']$ is a clique.

**Lemma 2.** *For all $2 \leq k \leq n-1$, hypergraph $H$ has a proper $k$-coloring if and only if $\mathbf{rvc}(G) \leq \mathbf{srvc}(G) \leq k$.*

Now, the stated theorem follows by arguments similar to those used in the proof of Theorem 1. The analogous results for $\mathbf{srvc}(G)$ follow by observing that the previous arguments also work for shortest paths. □

We conclude this section by showing positive results for some subclasses of chordal graphs. A graph $G$ is a *block graph* if every maximal 2-connected component of $G$ is a clique. A graph is an *interval graph* if it is an intersection graph of a set of intervals of the real line and it is a *unit interval graph* if all the intervals in the intersection model have length one.

**Theorem 4.** *Let $G$ be a block graph, and let $\ell$ be the number of cut vertices in $G$. Then $\mathbf{rvc}(G) = \mathbf{srvc}(G) = \ell$.*

**Theorem 5.** *If $G$ is an interval graph, then $\mathbf{rvc}(G) = \mathrm{diam}(G) - 1$.*

**Theorem 6.** *If $G$ is a unit interval graph, $\mathbf{rvc}(G) = \mathbf{srvc}(G) = \mathrm{diam}(G) - 1$.*

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