Graph searches and geometric convexities in graphs[1]

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Abstract

We discuss the relationship between graph searches and graph convexities. Nearly every graph search produces a convex geometry, i.e. every set of visited vertices is convex for some convex geometry. Many convex geometries have been defined on graphs using an interval definition: geodesic, monophonic .... We recall that the Lexicographic Breadth First Search (LexBFS) captures these convex geometries on many graph classes and we finish with some new properties on AT-free graphs.

1 Introduction

It is well-known that for various special graph classes, lexicographic breadth first search, LexBFS for short, always ends on an extreme point for a given convex geometry defined on these classes of graphs. For example on chordal graphs, LexBFS always ends at a simplicial vertex which is extreme for the monophonic convexity [10]. Indeed, in an attempt to understand why the orderings produced by this graph search and others (MNS, LexDFS, etc) have been so successful, one quickly notices that the orderings produced by these traversals are precisely words of some antimatroids or convex geometries. The notion of antimatroids and convex geometries have appeared in the literature under various settings; in this work, we focus on the graph searching setting, where we discuss some known geometries on cocomparability graphs, and then present new structural properties on AT-free graphs, a superclass of cocomparability graphs.

A *convexity space* is a tuple $(V, \mathcal{N})$, where V is a finite ground set, and $\mathcal{N}$ is a collection of subsets of V such that $\emptyset \in \mathcal{N}$, $V \in \mathcal{N}$, and $\mathcal{N}$ is closed under intersection. The elements of $\mathcal{N}$ are called convex sets. One can define a convex hull of a subset $S \subseteq V$ in the natural way as $\tau(S)$ being the intersection of all the convex sets that contain S. With every closure operator, it is natural to talk of *extreme points* as the set of points one can delete and still maintain a convex set. If a convexity space satisfies the following condition, known as *the anti-exchange property*, then it is called *a convex geometry*.

The Anti-Exchange Property: Let $S \subseteq V$ and $a, b, \in V$ such that $a, b \notin \tau(S)$. Then $a \in \tau(S \cup \{b\}) \implies b \notin \tau(S \cup \{a\})$.

Introduced first by Edelman and Jamison in 1985 [8], convex geometries have appeared in the literature under different names and aspects, the most famous one being *antimatroids*. Given a convex geometry $(V, \mathcal{N})$, then the set system $(V, \mathcal{F})$ where $\mathcal{F} = \{X : X = V \backslash Y, Y \in \mathcal{N})$ is an *antimatroid*. We call the sets $X \in \mathcal{F}$ feasible sets. Antimatroids capture various eliminations orderings on graph classes, which are the basis of a number of efficient algorithms. There is a close relationship between convexity spaces and the notion of *abstract betweenness* defined by Menger [11]. Betweenness is a ternary relation, that relates the "placement" of a point z between two other points a and b. We say (and write) z belongs to the interval $I[a, b]$ when z is between a and b. The notion of what an interval is varies with every betweennesses. As noticed by Chvátal [2], several abstract betweennesses can be defined on graphs. Given a betweenness relation, one can deduce a convexity as follows: $C \subseteq V(G)$ is convex if $\forall a, b \in C$ and $\forall z \in I[a, b]$, $z \in C$.

A well studied antimatroid is the one that rises from chordal graphs. These are graphs where the largest induced cycle is a triangle. Chordal graphs are characterized by a vertex ordering known as a *perfect*

[1] This is an extended abstract, all the proofs are omitted.

elimination ordering, or PEO. An ordering $\sigma = v_1, v_2, \ldots, v_n$ is a PEO if for all $i \in [n]$, v_i is simplicial in $\sigma[v_1, \ldots, v_{i-1}]$; Meaning the neighbourhood of v_i to its left in σ induces a clique. Thus v_n is a simplicial vertex in G. The set system $(V, \mathcal{F})$ whose ground set are the vertices of a chordal graph, and its feasible sets are the suffixes of PEOs form an antimatroid [10]. Given such an antimatroid, the corresponding convex geometry is the tuple $(V, \mathcal{N})$ where $S \subseteq V$ is convex if for all chordless paths between the vertices in S are also in S. That is, for all $a, c \in S$, if b lies in a chordless ac path in G, then $b \in S$. This is known as the monophonic convexity. Indeed the set system $(V, \mathcal{N})$ is a convex geometry if and only if G is chordal [9].

A second well studied antimatroid is the double shelling antimatroid of posets. Let $P(V, \prec)$ be a poset. The double shelling antimatroid on P is the set system whose feasible sets are unions of ideals and filters of P. It is easy to see that the corresponding convex geometry is a set system $(V, \mathcal{N})$, where a set $S \in \mathcal{N}$ is convex if for all $a, c \in V$, every b that satisfies $a \prec b \prec c$ or $c \prec b \prec a$ is also in S.

For a more comprehensive survey on this topic, we refer the reader to the monograph *Greedoids* by Korte, Lovász, and Schrader [10], and to [7, 9] for more on convexities in graph.

In this work we study convex geometries that rise from graph searches, in particular on graph searches when applied to cocomparability and AT-free graphs, using the following two convexities below. By convention, an extreme vertex for V, will be called an extreme vertex of G.

cc-convexity: Cocomparability convexity : $I_{cc}[x, y] = \{z | x, y, z$ is an independent triple and $\forall \sigma$ cocomparability ordering of G, either $x \prec_\sigma z \prec_\sigma y$ or $y \prec_\sigma z \prec_\sigma x\}$.

2p-convexity: 2-paths convexity: $I_{2p}[x, y] = \{z | x, y, z$ is an independent triple and $\exists P, Q$ two induced paths, such that P from z to x avoids $N[y]$, and Q from z to y avoids $N[x]\}$.

We present the following theorem, without proof, which connects these two type of convexities. Notice however, that the converse of the theorem is not true as can be seen with any long chain.

Theorem 1.1. *If G is a cocomparability graphs, then $z \in I_{2p}[x, y] \implies z \in I_{cc}[x, y]$.*

2 Graph searches and convexities

A graph search is a mechanism to traverse a graph one vertex at a time. Some well known graph searches are BFS and DFS. In this section we use the terminology defined in [4] for usual graph searches.

In this section, we focus on two graph classes: cocomparability graphs and asteroidal triple free graphs (or AT-free). This latter graph class contains cocomparability graphs. An asteroidal triple in a graph G is an independent triple of vertices x, y, z, where every pair of the triple is connected via a path that avoids the neighbourhood of the third vertex. An AT-free graph is a graph with no asteroidal triples.

Let $G(V, E)$ be an AT-free graph and σ a LexBFS of G. An ordering τ of G is an AT-free order of V if for all $z \in [x, y]$, $z \prec_\tau x$ or $z \prec_\tau y$. Two vertices u, v are **unrelated** with respect to a third vertex w if there exists a path P_{uw} that v does not see and a path P_{vw} that u does not see. A vertex w is **admissible** if no two vertices u, v are unrelated with respect to w. An ordering $\sigma = v_1 \ldots v_n$ is an admissible elimination ordering (AEO for short) of G if for all $i \in [n]$, v_i is admissible in $G[v_i \ldots v_n]$.

Definition 2.1. For a given graph property $\mathcal{P}$, we say that a vertex ordering $\tau = v_1, v_2, \ldots, v_n$ is $\mathcal{P}$-perfect if $\forall v_i, \mathcal{P}(v_i)$ is true for v_i in $G[v_i, \ldots, v_n]$. We say that $\mathcal{P}$ is hereditary if it satisfies the property that $\forall v \in V(G)$, and $\forall H$ subgraph of G, if $\mathcal{P}(v)$ holds in G, then if $v \in V(H)$, $\mathcal{P}(v)$ holds in H as well.

Using the above definition, we observe the following simple but key theorem, which generalizes the classical antimatroid of simplicial elimination schemes.

Theorem 2.2. *If $\mathcal{P}$ is a hereditary property of a graph class, then the set $\mathcal{S}$ of prefixes of $\mathcal{P}$-perfect orderings form an antimatroid $(V, \mathcal{S})$.*

Examples of Theorem 2.2 include property $P(x) = \{d(x) \leq k\}$ for instance, or $P(x) = \{x$ is not the middle vertex of a $P_4\}$, or $P(x) = \{x$ is a, true or false, twin$\}$. Let us consider a graph search $\mathcal{S}$, if we consider the sequence of visited vertices as follows: $V_0 = \{x_0\}, V_1 = \{x_0, x_1\}, \ldots V_n = \{V\}$. It yields a $\mathcal{S}$-convexity on $V(G)$ by defining $\forall A \subseteq V$, $conv(A) = V_i$, where i is the smallest index for which $A \subseteq V_i$.

Lemma 2.3. *If G is connected and $\mathcal{S}$ a generic search then its $\mathcal{S}$-convexity is a convex geometry.*

As a consequence, since complements of convex geometries are antimatroids, the suffixes of generic graph searches considered as sets form an antimatroid. Thus, the very natural question we are faced with is:

For which graph searches and graph classes is this $\mathcal{S}$-convexity an interesting convexity in graphs?

Lemma 2.4. *If at each step, the set of visited vertices V of a graph search $\mathcal{S}$ is a $\mathcal{C}$ convex set, then at each step the next vertex to be visited is an extreme vertex for V with respect to the convexity $\mathcal{C}$.*

There is no equivalence between these two above properties, but if the graph searches are restricted we have:

Theorem 2.5. *Let $\mathcal{S}$ be a generic search and $\mathcal{G}$ an hereditary class of graphs. The three following properties are equivalent for a graph convexity $\mathcal{C}$:*

(i) *At each step of the search $\mathcal{S}$ the set of visited vertices is a $\mathcal{C}$-convex set*

(ii) *The last vertex visited by $\mathcal{S}$ is extreme for the $\mathcal{C}$ convexity.*

(iii) *The suffixes of the search $\mathcal{S}$ considered as sets form an antimatroid.*

In this case, we say that $\mathcal{S}$ is **compatible** with the $\mathcal{C}$ convexity. As a consequence, for a hereditary class of graphs, to check the compatibility of a given graph search with a convexity, it is enough to test whether the last vertex is always extreme for this convexity. Let us now examine some properties for the graph convexities described above.

Lemma 2.6. *[5] If G is an AT-free graph, every LexBFS ends at an admissible vertex of G.*

Although there are non-AT-free graphs where LexBFS produces an AEO, Corneil and Köhler noticed that:

Theorem 2.7. *[3] Every LexBFS search is compatible with the 2p-convexity iff G is AT-free.*

Notice that the existence of one single admissible elimination ordering does not characterize AT-free graphs –as can be shown in Fig. 2. Both graphs contain an asteroidal triple, but both admit an admissible elimination scheme. Moreover for the rocket graph, its unique extreme point cannot be the end of a LexBFS. So if we want to use this theorem to build a recognition algorithm for AT-free graphs, we need to check for all LexBFSs.

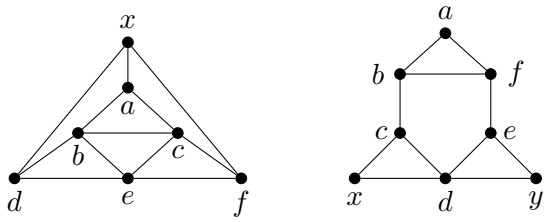


Figure 1: A sun$^+$ and a rocket

2.1 New Properties on AT-free graphs

We next give a few properties on AT-free graphs by looking at the underlying convex geometry structure of this class. Consider the set family $\mathcal{F}$ of sets of vertices that are prefixes of AEOs of G AT-free. We have the following obvious property:

Property 1. The set system $(V, \mathcal{F})$ forms an antimatroid.

Independently, Chang et al. proved a stronger result that shows that the 2p-convexity is a convex geometry on AT-free graphs [1]. This is stronger than Property 1 since it exhibits the type of convexity of AT-free graphs, and in particular the type of betweenness of this graph class.

Lemma 2.8. *Let x be a LexBFS end vertex of G, thn there exists an AT-free ordering τ of G where x is the end vertex of τ.*

Theorem 2.9. *G is AT-free iff every induced subgraph H has ≥ 2 extreme vertices for the 2p-convexity which are ends of LexBFS.*

Theorem 2.10. *Every AT-free graph is either a clique or has ≥ 3 admissible vertices*

This latter theorem is reminiscent of the classical property on chordal graphs by Dirac, which states that every chordal graph is either a clique or has at least two non-adjacent simplicial vertices [6].

Conclusion: Can we use the antimatroid structure of AT-free graphs to lift some algorithms from cocomparability graphs to AT-free graphs?

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