Cycle creating paths

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Abstract

We say that two paths on the same vertex set create a k-cycle if the union of the two paths (the union of their edges) contains a k-cycle. Let the maximal number of pairwise k-cycle creating paths on n vertices be $H(n,k)$. We study $H(n,k)$ for fixed k and large n. The behavior of $H(n,k)$ when k is even is very different than when k is odd. The only known exact value of $H(n,k)$ is when $k=3$. In this talk we present the known bounds involving $H(n,k)$ and we discuss the connection between $H(n,3)$ and the colliding permutations conjecture, and the connection between $H(n,4)$ and the reversing permutations conjecture.

The notation $H(n,k)$ is motivated by the observation that given a set of C_k-creating paths (of unspecified length) on n vertices, if one of the paths does not use all the vertices, we can lengthen it arbitrarily without destroying the C_k-creating property of the set. Therefore from now on, we assume that every path is of maximal length. (A Hamiltonian path of the complete graph K_n, hence the H.) The behavior of $H(n,2k)$ and $H(n,2k+1)$ is very different.

1 Odd cycles

Proposition 1.1. *$H(n,2k+1)$ is at most the number of balanced bipartitions of $[n]$.*

Proof. Since every path is a bipartite graph and for every bipartition of the ground set we can choose only a single path with this bipartition. Otherwise, the union of the two would still be bipartite contradicting that they are C_{2k+1}-creating. □

In [6] Körner, Messuti and Simonyi asked whether $H(n,3)$ is actually equal to the number of balanced bipartitions of $[n]$. This was positively answered in [4].

Theorem 1.2 (I. Kovács, D. S.)**.** [4]

$$H(n,3)=\begin{cases}\binom{n}{\lfloor\frac{n}{2}\rfloor} & \text{when } n\equiv 1 \mod 2\\ \frac{1}{2}\binom{n}{\lfloor\frac{n}{2}\rfloor} & \text{when } n\equiv 0 \mod 2.\end{cases}$$

The lower bound of Theorem 1.2 is constructive but much longer than the proof of the upper bound. The construction method of Theorem 1.2 was generalized to give the following asymptotic result.

Theorem 1.3 (I. Kovács, D. S.)**.** [3] *For every positive integer ℓ,*

$$H(n,2^\ell+1)=2^{n-o(n)}.$$

This led us to formulate the following conjecture.

Conjecture 1.4. [3] *For all positive integers k, $H(n,2k+1)=2^{n-o(n)}$.*

Conjecture 1.4 is still open, there is some asymptotic supporting evidence for it.

Theorem 1.5 (I. Kovács, D. S.)**.** [3]

$$\lim_{k\to\infty}\lim_{n\to\infty}\sqrt[n]{H(n,2k+1)}=2.$$

The original motivation of Körner, Messuti and Simonyi was the so-called *colliding* permutations conjecture. Two permutations – viewed as vectors of length n consisting of the elements of $[n]$ – are colliding if there is a coordinate where their difference is exactly 1.

Conjecture 1.6. [5] *The maximal number of pairwise colliding permutations is* $\binom{n}{\lfloor n/2\rfloor}$.

The problem of determining the maximal number of pairwise colliding permutations and the problem of determining $H(n,3)$, feels somewhat similar. This feeling of similarity can be justified as follows.

Let $GH(n,3)$ be a graph whose vertices are the Hamiltonian paths of K_n and two vertices are connected if the corresponding Hamiltonian paths are C_3-creating. Clearly, $H(n,3)$ is the clique number of $GH(n,3)$. Let $GC(n)$ be a graph whose vertices are the permutations of $[n]$ and two vertices are connected when the corresponding permutations are colliding. Clearly, the maximal number of pairwise colliding permutations is the clique number of $GC(n)$.

Proposition 1.7. *$GC(n)$ is an induced subgraph of $GH(2n,3)$.*

2 Even cycles

As stated before, the behavior of $H(n,2k)$ is very different than the behavior of $H(n,2k+1)$.

Proposition 2.1.

$$n^{\frac{1}{k}n-o(n)}\le H(n,2k).$$

The statement of Proposition 2.1 can be proven for $H(n,4)$ as follows. Let $\mathcal{H}$ be the set of paths that start at vertex $1\in[n]$ and for every $i\in\{0,\lfloor n/2\rfloor\}$ the $(2i+1)$-th vertex of the path is $2i+1\in[n]$. In other words, the position of the even and odd numbers from $[n]$ is the same in every path, and the position of every odd element is the same, but the even elements can vary among themselves. Thus to every path in $\mathcal{H}$ corresponds a permutation of the even elements in $[n]$. Two such paths are C_4-creating since there must be two odd elements of $[n]$ where between these two our paths differ. But then the two odd elements and the two even ones that are between them form a C_4 in the union of the two paths. The size of $\mathcal{H}$ is roughly $(n/2)!=n^{n/2-o(n)}$.

A similar construction works for longer even cycles. The best known upper bounds were proven by Cohen, Fachini and Körner for $H(n,4)$.

Theorem 2.2 (G. Cohen, E. Fachini, J. Körner)**.** [2]

$$H(n,4)\le n^{\frac{3}{4}n}$$

This was generalized in [7] for $H(n,2k)$ in general.

Theorem 2.3 (D. S.)**.** [7]

$$H(n,2k)\le n^{\left(1-\frac{2}{3k^2-2k}\right)n}$$

Thus unlike in the case of odd cycles, our bounds are getting worse and worse while k grows. There is no conjectured growth rate for $H(n, 2k)$ in general, but the following conjecture of Körner is related to $H(n, 4)$.

We say that two permutations π_1, π_2 are reversing if there are two elements $a, b \in [n]$ which occupy the same two coordinates in π_1 and π_2 but in one of the permutations a is preceding b and in the other permutation b precedes a.

Conjecture 2.4 (Körner)**.** *The maximal number of pairwise reversing permutations of* $[n]$ *is less than* C^n *for some real* C*.*

If Conjecture 2.4 is true then $H(n, 4) = n^{\frac{1}{2}n - o(n)}$. It is not hard to prove that the maximal number of pairwise reversing permutations of $[n]$ is just a polynomial factor away from the maximal number of C_4-creating perfect matchings on $2n$ vertices. This makes the connection between $H(n, 4)$ and the maximal number of pairwise reversing permutations more natural. The connection between $H(n, 2k)$ and the maximal number of pairwise C_{2k}-creating perfect matchings for $k > 2$ is even stronger. If our upper bounds on the maximal number of pairwise C_{2k}-creating perfect matchings are improved, our upper bound (in Theorem 2.3) on $H(n, 2k)$ improves as well, and if our lower bounds on C_{2k} are improved, our lower bound (in Proposition 2.1) on $H(n, 2k)$ improves as well.

References

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