Structural characterizations for polynomially-time recognizable (r, ℓ)-well covered graphs.

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Abstract

An (r, ℓ)-partition of a graph G is a partition of its vertex set into r independent sets and ℓ cliques. A graph is (r, ℓ) if it admits an (r, ℓ)-partition. The dichotomy P versus NP-complete for recognizing (r, ℓ)-graphs is well-known [2]: the problem is in P if $\max\{r, \ell\} \leq 2$, and NP-complete otherwise. Well-covered graphs are graphs in which every maximal independent set is also maximum. The problem of recognizing a well-covered graph was proved to be coNP-complete [4, 10]. A graph G is said (r, ℓ)-well-covered if G is both (r, ℓ) and well-covered. In (r, ℓ)-WELL-COVERED ((r, ℓ)WCG) problem, we are given an undirected graph $G = (V, E)$, and we are asked to determine whether G is a (r, ℓ) and a well-covered graph. Alves et al. proved in [1] that (r, ℓ)-WELL-COVERED problem is polynomial when $r \leq 1$ and $\ell \leq 2$ or $r = 2$ and $\ell = 0$, and hard otherwise. In this paper we give structural characterizations to each of the following six subclasses: $(0, 1)$WCG, $(0, 2)$WCG, $(1, 0)$WCG, $(1, 1)$WCG, $(1, 2)$WCG, and $(2, 0)$WCG.

1 Introduction

An (r, ℓ)-*partition* of a graph $G = (V, E)$ is a partition of V into r independent sets $S^1, \ldots, S^r$ and ℓ cliques $K^1, \ldots, K^\ell$. We observe that some of these sets can be empty. A graph is (r, ℓ) if it admits an (r, ℓ)-partition. The dichotomy P versus NP-complete for recognizing (r, ℓ)-graphs is well-known [2]: the problem is in P if $\max\{r, \ell\} \leq 2$, and NP-complete otherwise. The class of (r, ℓ)-graphs and its subclasses have been extensively studied in the literature. For instance, list partitions of (r, ℓ)-graphs, chordal- (r, ℓ)-graphs, and cographs-(r, ℓ) were studied in [5, 6, 3], respectively.

A graph G is called *well covered* if every two maximal independent sets of G have the same cardinality. In other words, a graph G is well covered if every maximal independent set of G is a maximum independent set of G. The concept of well covered graph was introduced by Plummer [8]. Since the problem of finding the independence number of a general graph is NP-complete and, once applying the greedy algorithm for producing a maximal independent set always produces a maximum independent set when applied to well covered graphs, this class of graphs turns out to have interesting algorithmic properties. The problem of recognizing a well-covered graph, which we denote by WELL-COVERED GRAPH, was proved to be coNP-complete by Chvátal and Slater [4] and independently by Sankaranarayana and Stewart [10], but is in P when the input graph is known to be, for instance, a cograph [7]. One of the main directions of the work on well covered graphs has been done towards characterizing some of their subclasses, exploring estructural properties of these classes. In this paper we are interested in characterizing graphs that are both, (r, ℓ) and well-covered, and can be recognized in polynomial time.

2 Complexity of (r, ℓ)-well-covered graph problem

The decision problem (r, ℓ)-WELL-COVERED GRAPH is defined as follows:

(r, ℓ)-WELL-COVERED GRAPH ((r, ℓ)WCG)
Instance: A graph $G = (V, E)$.
Question: Is G (r, ℓ) and well-covered?

The complexity of this decision problem was established in [1] and it is shown in Table 1.

(r, ℓ)WCG	0	1	2	≥ 3
0	-	P	P	NPc
1	P	P	P	NPc
2	P	coNPc	coNPc	(co)NPh
≥ 3	NPh	(co)NPh	(co)NPh	(co)NPh

Table 1: Complexity of the decision problem (k, ℓ)WCG.

Hence, there are six polynomially-time recognizable subclasses of (r, ℓ)WCG: $(1, 0)$WCG, $(0, 1)$WCG, $(1, 1)$WCG, $(1, 2)$WCG, $(2, 0)$WCG and $(0, 2)$WCG. In Section 3, we give structural characterizations for all these subclasses. Note that there are some pairs (r, ℓ) of the decision problem (r, ℓ)WCG which belong, at the same time, to both classes NP-hard and coNP-hard. This is a consequence of two facts: the recognition problem of a (r, ℓ)-graph, when $r > 2$ or $\ell > 2$, is NP-complete (see [2]), and the recognition problem of well-covered graphs, for graphs in general, is coNP-complete (see [4]).

3 Structural characterizations

In this section we present characterizations for the six polynomially-time recognizable (r, ℓ)WCG subclasses. Our main result is the characterization of the $(1, 2)$-well covered graph.

Fact 1. *A graph G whose $V(G)$ induces either a clique or an independent set is well-covered.*

The following Corollary is a simple application of Fact 1.

Corollary 1. *G is a $(0, 1)$-WCG if, and only if, G is a $(0, 1)$-graph. Similarly, G is a $(1, 0)$WCG if and only if G is a $(1, 0)$-graph.*

Theorem 1 provides a characterization for the decision problem $(0, 2)$WCG.

Theorem 1. *Let $G = (V, E)$ be a graph. G is a $(0, 2)$WCG if, and only if, either G is a complete graph or G is a co-bipartite graph having no universal vertex.*

In [1] we give a necessary and sufficient condition for a graph to be a $(1, 1)$-WCG. In the literature, $(1, 1)$-graphs are often called *split graphs*.

Theorem 2. *[1] A graph $G = (V, E)$ is a $(1, 1)$WCG if, and only if, it admits a $(1, 1)$ partition (S, K) such that either, $\forall v \in K$, $|N_S(v)| = 0$, or, $\forall v \in K$, $|N_S(v)| = 1$.*

The structural characterization of a $(2, 0)$-WCG is a result of Ravindra [9] and it is described in Theorem 3.

Theorem 3. *[9] Let $G = (V, E)$ be a graph. G is a $(2, 0)$-*WCG *if, and only if, G contains a perfect matching F such that, for every edge $e = uv$ in F, $G[N(u) \cup N(v)]$ is a complete bipartite graph.*

Next, we present a structural characterization of $(1, 2)$WCG.

Theorem 4. *Let $G = (V, E)$ be a graph with partition $V = (S, K^1, K^2)$ where S is a maximal independent set, and K_1, K_2 are cliques.Then, $G = (V, E)$ is a $(1, 2)$-well covered graph if and only if G satisfies the four conditions:*

1. *If $v \in K^1 \cup K^2$, then $1 \leq |N_S(v)| \leq 2$,*
2. *If $u, v \in K^1 \cup K^2$ with $uv \notin E$, then $1 \leq |N_S(v) \cup N_S(u)| \leq 2$,*
3. *If $\exists v \in K^i$ such that $|N_S(v)| = 2$, then $\exists u \in K^j, i \neq j$ such that $uv \notin E$ and $N_S(u) \subseteq N_S(v)$,*
4. *Given $v \in K^i$ with $|N_S(v)| = 1$. If $u \in K^j, i \neq j$ with $uv \notin E$, then $|N_S(u) \cup N_S(v)| = 2$.*

Proof. Suppose G a $(1, 2)$–well covered graph with a partition $V = (S, K^1, K^2)$ where S is maximal independent set. We prove the necessity of conditions 1, 2, 3 and 4 bellow.

(1.) First, note that $|N_S(v)| \geq 1, \forall v \in (K^1 \cup K^2)$, otherwise S would not be maximal. Next, suppose by contradiction, there is a vertex $v \in K^1 \cup K^2$, such that $|N_S(v)| \geq 3$. We observe that $|S| - 1$ is an upper bound to the size of a maximal independent set containing v, since the set $I = \{v\} \cup S \setminus N_S(v) \cup \{w\}$, where w is a vertex of K^2 possibly non adjacent to a vertex of $\{v\} \cup S \setminus N_S(v)$, reaches this upper bound ($|I| \leq 1 + |S| - 3 + 1 = |S| - 1$). Thus, G is not well covered, once S and I are maximal independent sets of different sizes. A contradiction. Then, $\forall v \in K^1 \cup K^2$, $|N_S(v)| \leq 2$.

(2.) Suppose, by contradiction, there is a pair of vertices $u \in K^1$ and $v \in K^2$, such that $uv \notin E$, and $|N_S(u) \cup N_S(v)| \geq 3$. We observe that, again, $|S| - 1$ is an upper bound for an independent set containing v, since the set $I = \{u, v\} \cup S \setminus (N_S(u) \cup N_S(v))$ reaches this upper bound ($|I| \leq 2 + |S| - 3 = |S| - 1$). Hence, G is not well covered. A contradiction. Then, $\forall u, v \in K^1 \cup K^2$, $uv \notin E$, $1 \leq |N_S(v) \cup N_S(u)| \leq 2$.

(3.) Assume that $v \in K^1$ with $|N_S(v)| = 2$. If there is no vertex $u \in K^2$ with $uv \notin E$, then we have that v is adjacent to every vertex of K^2. Hence, $\{v\} \cup S \setminus N_S(v)$ is a maximal independent set of size $1 + |S| - 2 = |S| - 1$, a contradiction.
Thus, suppose, by contradiction, there is a vertex $u \in K^2$ with $uv \notin E$, such that $N_S(u) \not\subseteq N_S(v)$. Since $|N_S(v)| = 2$ and S is maximal, we have that $|N_S(u) \cup N_S(v)| \geq 3$, what contradicts condition 2. Hence, $\forall v \in K^2$ with $|N_S(v)| = 2$, $\exists u \in K^2$ such that $uv \notin E$ and $N_S(u) \subseteq N_S(v)$.

(4.) Let $v \in K^1$ with $|N_S(v)| = 1$. Suppose there exists $u \in K^2$ such that $uv \notin E$ and, by contradiction, suppose $|N_S(u) \cup N_S(v)| \neq 2$. It follows from condition 2 that $|N_S(u) \cup N_S(v)| = 1$, since $1 \leq |N_S(u) \cup N_S(v)| \leq 2$. Hence, the maximal independent set $\{u, v\} \cup S \setminus N_S(u)$ has size $2 + |S| - 1 = |S| + 1$ contradicting the fact that G is well covered. Hence, $\forall v \in K^1$ with $|N_S(v)| = 1$, if there exists $u \in K^2$ such that $uv \notin E$, then $|N_S(u) \cup N_S(v)| = 2$.

Now suppose the graph G satisfies the conditions 1, 2, 3 and 4 listed above.

Observe that the maximal independent sets of G can be partitioned into three types: those who have no vertex in $K^1 \cup K^2$, those who have exactly one vertex in $K^1 \cup K^2$, or those who have exactly two vertices in $K^1 \cup K^2$.
Next, we prove that, under the assumptions 1, 2, 3 and 4, all maximum independent sets of G

have size $|S|$. Let I be a maximal independent set containing exactly a vertex v in $K^1 \cup K^2$, say $v \in K^1$. From condition 1 and from the fact that S is maximal, we know that, either, $|N_S(v)| = 2$ or $|N_S(v)| = 1$.
Suppose $|N_S(v)| = 2$. From condition 3 we have that a maximal independent set I containing v must contain a vertex $u \in K^2$ such that $N_S(u) \subseteq N_S(v)$. In this case, $I = \{v, u\} \cup S \setminus N_S(v)$ and $|I| = 2 + |S| - 2 = |S|$.
Now, suppose $|N_S(v)| = 1$ and let I be a maximal independent set containing v. By condition 4, we have two cases to analyze: when v is adjacent to all vertices of K^2, and when there is a vertex $u \in K^2$ such that $uv \notin E$. In the first case, $I = \{v\} \cup S \setminus N_S(v)$ and then $|I| = 1 + |S| - 1 = |S|$. In the second case, we can have two additional cases: the case when $I = \{v\} \cup S \setminus N_S(v)$, and the case when $I = \{v, u\} \cup S \setminus (N_S(v) \cup N_S(u))$, for some vertex $u \in K^2$ with $uv \notin E$. In the first case, notice that I is maximal since that, by condition 4, every vertex of K^2 nonadjacent to v has a neighbor in $S \setminus N_S(v)$. Thus, $|I| = 1 + |S| - 1 = |S|$. In the second case, again by condition 4., we have that $|S \setminus (N_S(v) \cup N_S(u))| = |S| - 2$. Hence, $|I| = 2 + |S| - 2 = |S|$.
Therefore, we conclude that every maximal independent set of G has size $|S|$, and, then, G is well covered. □

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