Equitable Coloring of subclasses of Chordal Graphs

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Abstract

A graph on n vertices is equitably k-colorable if it is k-colorable and every color is used either $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ times. Such a problem appears to be considerably harder than vertex coloring, being NP-Complete even for cographs and interval graphs. In this work, we prove that it is W[1]-Hard for block graphs when parameterized by the number of colors; and for $K_{1,4}$-free interval graphs when parameterized by treewidth, number of colors and maximum degree, generalizing a result by Fellows et al. (2014) through a much simpler reduction. Finally, using a previous result due to de Werra (1985), we establish a dichotomy for the complexity of equitable coloring of chordal graphs based on the size of the greatest induced star.

1 Introduction

EQUITABLE COLORING is a variant of the classical VERTEX COLORING problem, where we not only want to partition an n vertex graph into k independent sets, but also that each of these sets has either $\lfloor n/k \rfloor$ ou $\lceil n/k \rceil$ vertices. The smallest integer k for which G admits an equitable k-coloring is called the *equitable chromatic number* of G.

An extensive survey was conducted by Lih [12], where many of the results on EQUITABLE COLORING of the last 50 years were assembled. Most of them, however, are upper bounds on the equitable chromatic number. Such bounds are known for: bipartite graphs, trees, split graphs, planar graphs, outerplanar graphs, low degeneracy graphs, Kneser graphs, interval graphs, random graphs and some forms of graph products.

Almost all complexity results for EQUITABLE COLORING arise from a related problem, known as BOUNDED COLORING, an observation given by Bodlaender and Fomin [2]. On BOUNDED COLORING, we ask that the size of the independent sets be bounded by an integer l, which is not necessarily a function on k or n. Among the known results for BOUNDED COLORING, we have that the problem is solvable in polynomial time for: split graphs [4], complements of interval graphs [3], forests [1], trees [11] and complements of bipartite graphs [3]. For cographs, there is a polynomial-time algorithm when the number of colors k is fixed, otherwise the problem is NP-Complete [3]; the same is also valid for bipartite graphs and interval graphs [3]. A consequence of the difficulty of BOUNDED COLORING for cographs is the difficulty of the problem for graphs of bounded cliquewidth. In complements of comparability graphs, even if we fix l, BOUNDED COLORING remains NP-Hard. In [7], it is shown that EQUITABLE COLORING parameterized by treewidth and number of colors is W[1]-Hard and, in [2], an XP algorithm for graphs of bounded treewidth is given.

In this work, we perform a series of reductions proving that EQUITABLE COLORING is W[1]-Hard for different subclasses of chordal graphs. In particular, we show that the problem parameterized by the number of colors is W[1]-Hard for block graphs and for the disjoint union of split graphs. Moreover, the problem remains W[1]-Hard for $K_{1,4}$-free interval graphs even if we parameterize it by treewidth, number of colors and maximum degree. This last results generalize the proof given by Fellows et al. in [7] that EQUITABLE COLORING is W[1]-Hard when parameterized by treewidth and number of colors. A result given by de Werra in [13] guarantees that every $K_{1,3}$-free graph

can be equitable k-colored if k is at least the chromatic number. Since VERTEX COLORING can be solved in polynomial time on chordal graphs, we trivially have a polynomial-time algorithm for EQUITABLE COLORING of $K_{1,3}$-free chordal graphs. This allows us to establish a dichotomy for the computational complexity of EQUITABLE COLORING of chordal graphs based on the size of the largest allowed induced star.

2 Preliminaries

We used standard graph theory notation. Define $[k] = \{1, \ldots, k\}$ and 2^S the *powerset* of S. A *k-coloring* φ of a graph G is a function $\varphi : V(G) \mapsto [k]$. Alternatively, a k-coloring is a k-partition $V(G) \sim \{\varphi_1, \ldots, \varphi_k\}$ such that $\varphi_i = \{u \in V(G) \mid \varphi(u) = i\}$. A set $X \subseteq V(G)$ is *monochromatic* if $\left|\bigcup_{u \in X}\{\varphi(u)\}\right| = 1$. A k-coloring is said to be *equitable* if, for every $i \in [k]$, $\lfloor n/k \rfloor \leq |\varphi_i| \leq \lceil n/k \rceil$. A k-coloring of G is *proper* if no edge of G is monochromatic, that is, if φ_i is an independent set for every $i \in [k]$. We only deal with proper colorings in this work.

The *disjoint union*, or simply *union*, of two graphs $G \cup H$ is a graph such that $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. The *join* of two graphs $G \oplus H$ is the graph given by $V(G \oplus H) = V(G) \cup V(H)$ and $E(G \oplus H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$. A graph is a *block graph* if and only if every biconnected component is a clique. A graph is a *split graph* if and only if $V(G)$ can be partitioned in a clique and an independent set.

A *tree decomposition* of a graph G is a pair $\langle \mathcal{T}, \mathcal{X} = \{X_j \mid j \in V(\mathcal{T})\}\rangle$, where $\mathcal{T}$ is a tree and $\mathcal{X} \subseteq 2^{V(G)}$ is a family where: $\bigcup_{X_j \in \mathcal{X}} X_j = V(G)$; for every edge $uv \in E(G)$ there is some X_j such that $\{u, v\} \subseteq X_j$; for every $i, j, q \in V(\mathcal{T})$, if q is in the path between i and j in $\mathcal{T}$, then $X_i \cap X_j \subseteq X_q$. Each $X_j \in \mathcal{X}$ is called a *bag*. The *width* of a tree decomposition is defined as the size of a largest bag minus one. The *treewidth* of a graph G is the smallest width among all valid tree decompositions of G [5].

3 Reductions

All of our reductions involve the BIN-PACKING problem, which is NP-Hard in the strong sense [9]. In the general case, the problem is defined as: given a set of positive integers $A = \{a_1, \ldots, a_n\}$ and two integers k and B, can we partition A into k *bins* such that the sum of the elements of each bin is at most B? We shall use a version of BIN-PACKING where each bin sums *exactly* to B. This second version is equivalent to the first, even from the parameterized point of view; it suffices to add $kB - \sum_{j \in [n]} a_j$ unitary items to A. BIN-PACKING is known to be W[1]-Hard when parameterized by the number of bins [10]. By BIN=PACKING we shall refer to the second version.

The idea for the following reductions is to build one gadget for each item a_j of the given BIN-PACKING instance, unite them, and equitably k-color the resulting graph. The color given to the circled vertices in Figure 1 control the bin to which the corresponding item. Each reduction uses only one of the three gadget types. Since every gadget is a chordal graph, their treewidth is precisely the size of the largest clique minus one, that is, k, which is also the number of desired colors for the built instance of EQUITABLE COLORING. For $K_{1,4}$-free interval graphs, note that the maximum degree is $3(k-1)$.

Definition 1. An (a, k)-antiflower is the graph $F_{-}(a, k) = K_{k-1} \oplus \left(\bigcup_{i \in [a+1]} K_1\right)$, that is, it is the graph obtained after performing the disjoint union of $a + 1$ K_1's followed by the join with K_{k-1}.

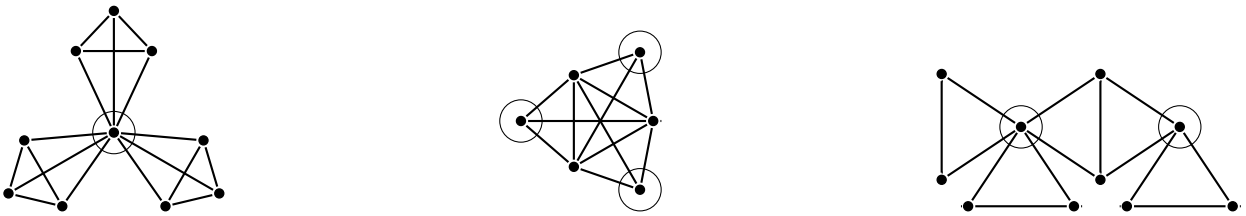


Figure 1: A $(2,4)$-flower, a $(2,4)$-antiflower, and a $(2,2)$-train.

Theorem 1. EQUITABLE COLORING *of the disjoint union of split graphs parameterized by the number of colors is* W[1]-Hard.

Proof. Let $\langle A, k, B\rangle$ be an instance of BIN-PACKING and G a graph such that $G = \bigcup_{j\in[n]} F_-(a_j + 1, k)$. Note that $|V(G)| = \sum_{j\in[n]} |F_-(a_j+1,k)| = \sum_{j\in[n]} k + a_j = nk + kB$. Therefore, in any equitable k-coloring of G, each color class has $n + B$ vertices. Define $F_j = F_-(a_j + 1, k)$ and let C_j be the corresponding K_{k-1}. We show that there is an equitable k-coloring ψ of G if and only if $\varphi = \langle A, k, B\rangle$ is a YES instance of BIN-PACKING.

Let φ be a solution to BIN-PACKING. For each $a_j \in A$, we do $\psi(C_j) = [k] \setminus \{i\}$ if $a_j \in \varphi_i$. We color each vertex of the independent set of F_j with i. Thus, $|\psi_i| = \sum_{j|a_j\in\varphi_i} a_j + 1 + \sum_{j|a_j\notin\varphi_i} 1 = \sum_{j|a_j\in\varphi_i} a_j + 1 + \sum_{j\in[n]} 1 - \sum_{j|a_j\in\varphi_i} 1 = n + B$.

Now, let ψ be an equitable k-coloring of G. Note that $|\psi_i| = n + B$ and that the independent set of an antiflower is monochromatic. For each $j \in [n]$, $a_j \in \varphi_i$ if $i \notin \psi(C_j)$. That is, $n + B = |\psi_i| = \sum_{j|i\notin C_j} a_j + 1 + \sum_{j|i\in C_j} 1 = \sum_{j|i\notin C_j} a_j + 1 + \sum_{j\in[n]} 1 - \sum_{j|i\notin C_j} 1 = \sum_{j|i\notin C_j} a_j + n$, from which we conclude that $\sum_{j|i\notin C_j} a_j = B$. □

Definition 2. An (a,k)-flower is the graph $F(a,k) = K_1 \oplus \left(\bigcup_{i\in[a+1]} K_{k-1}\right)$, that is, it is obtained from the union of $a+1$ cliques of size $k-1$ followed by a join with K_1.

Theorem 2. EQUITABLE COLORING *of block graphs parameterized by the number of colors is* W[1]-Hard.

Definition 3. Let $\mathcal{Q} = \{Q_1, Q'_1, \ldots, Q_a, Q'_a\}$ such that $Q_i \simeq Q'_i \simeq K_{k-1}$ and $Y = \{y_1, \ldots, y_a\}$ be a set of vertices. An (a,k)-train is the graph $H(a,k)$ where $V(H(a,k)) = \mathcal{Q} \cup Y$ and $E(H(a,k)) = E\left(\bigcup_{i\in[a]}(Q_i \cup Q'_i) \oplus y_i\right) \cup E\left(\bigcup_{i\in[a-1]} y_i \oplus Q_{i+1}\right)$.

Theorem 3. *Let G be a $K_{1,r}$-free interval graph. If $r \geq 4$,* EQUITABLE COLORING *of G parameterized by treewidth, number of colors and maximum degree is* W[1]-Hard. *Otherwise, the problem is solvable in polynomial time.*

4 Future Work

In this work, we proved that, even for different subclasses of chordal graphs, EQUITABLE COLORING is a hard problem, even if we parameterize by treewidth, number of colors and maximum degree. This difficulty suggests the investigation of the dual problem. If $k \geq n/2$, checking for the existence of an equitable $(n-k)$-coloring can be solved using a maximum matching algorithm [6] on the complement of the input graph. If k is large enough, we have that EQUITABLE COLORING is equivalent to find k

disjoint triangles on the complement graph such that the other $n - 3k$ vertices can be perfectly matched. We believe this problem is of interest to the community by itself, since it generalizes both MAXIMUM MATCHING and TRIANGLE PACKING, which is in FPT when parameterized by k [8].

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