Bounding χ by a fraction of Δ for graphs without large cliques

Marthe Bonamy — **CNRS, LaBRI, Université de Bordeaux**
Tom Kelly — **University of Waterloo**
Peter Nelson — **University of Waterloo**
Luke Postle — **University of Waterloo**

Abstract

In this talk, I will present new results on coloring graphs of bounded maximum degree Δ with no cliques of size greater than ω. In the 90s, Johansson proved that $\chi = O(\Delta/\ln(\Delta))$ and $\chi = O(\Delta \ln(\ln(\Delta))/\ln(\Delta))$ if $\omega = 2$ and if ω is fixed, respectively. Reed's Conjecture states that $\chi \leq \lceil(\Delta + 1 + \omega)/2\rceil$. It is natural to ask if Reed's conjectured bound can be improved if $\omega = o(\Delta)$ when Johansson's results do not apply. We answer this question by proving that

$$\chi = O\left(\Delta\sqrt{\frac{\ln(\omega)}{\ln(\Delta)}}\right).$$

For sufficiently large Δ, our result implies that if $\omega^{(72c)^2} \leq \Delta$ then $\chi \leq \Delta/c$.

This bound actually holds for list-coloring and even correspondence-coloring (a.k.a. DP-coloring). In fact, we prove what we call a "local version" of it, which simultaneously implies our bound and the two aforementioned results of Johansson.

1 Motivation

In 1996, Johansson [9] famously proved that if G is a triangle-free graph of maximum degree at most Δ, then $\chi_\ell(G) = O\left(\frac{\Delta}{\ln(\Delta)}\right)$, where $\chi_\ell(G)$ is the *list-chromatic number* of G. Determining the best possible value of the leading constant in this bound is of general interest. The best known lower bound, using Δ-regular graphs, is $\frac{\Delta}{2\ln(\Delta)}$. In 1995, Kim [11] proved that the upper bound holds with a leading constant of $1+o(1)$ for graphs of girth at least five. In 2015, Pettie and Su [14] improved the leading constant in the upper bound for triangle-free graphs to $4+o(1)$, and in 2017, Molloy [13], in the following theorem, improved it to $1 + o(1)$, matching the bound of Kim.

Theorem 1.1 ([13])**.** *If G is a triangle-free graph of maximum degree at most Δ, then*

$$\chi_\ell(G) \leq (1 + o(1))\frac{\Delta}{\ln(\Delta)}.$$

Johansson [10] also proved that for any fixed $\omega \geq 4$, if G is a graph of maximum degree at most Δ with no clique of size ω, then $\chi_\ell(G) = O\left(\frac{\Delta \ln(\ln(\Delta))}{\ln(\Delta)}\right)$; however, the proof was never published. Molloy [13] proved the following stronger result, which holds even when ω is not fixed.

Theorem 1.2 ([13])**.** *If G is a graph of maximum degree at most Δ with no clique of size greater than ω, then*

$$\chi_\ell(G) \leq 200\omega\frac{\Delta \ln(\ln(\Delta))}{\ln(\Delta)}.$$

In 1998, Reed [15] conjectured the following, sometimes referred to as "Reed's ω, Δ, χ Conjecture."

Conjecture 1.3 ([15]). *If G is a graph of maximum degree at most Δ with no clique of size greater than ω, then*

$$\chi(G) \leq \left\lceil \tfrac{1}{2}(\Delta + 1 + \omega) \right\rceil.$$

It is possible that Conjecture 1.3 is also true for the list-chromatic number. As evidence for his conjecture, Reed [15] proved the following.

Theorem 1.4 ([15]). *There exists $\varepsilon > 0$ such that the following holds. If G is a graph of maximum degree at most Δ with no clique of size greater than ω, then*

$$\chi(G) \leq \lceil (1-\varepsilon)(\Delta+1) + \varepsilon\omega \rceil.$$

Note that Theorem 1.4 holds for $\varepsilon = \frac{1}{2}$ if and only if Conjecture 1.3 is true. In 2016, Bonamy, Perrett, and Postle [5] proved that Theorem 1.4 holds for $\varepsilon = \frac{1}{26}$ when Δ is sufficiently large. In 2017, Delcourt and Postle [6] proved that Theorem 1.4 holds for the list-chromatic number for $\varepsilon = \frac{1}{13}$ when Δ is sufficiently large. Results from Ramsey Theory imply that Theorem 1.4 is not true for any value of $\varepsilon > \frac{1}{2}$; for example, Spencer [16] showed the existence of a graph on n vertices with independence number 2 (and thus chromatic number at least $n/2$) such that every clique has size at most $n^{\frac{1}{2}+o(1)}$.

Theorem 1.2 implies Conjecture 1.3 when $\omega = o\left(\frac{\ln(\Delta)}{\ln(\ln(\Delta))}\right)$. It is natural to ask if a bound stronger than that of Conjecture 1.3 can be proved if $\omega = o(\Delta)$ even if $\omega \geq \frac{\ln(\Delta)}{200\ln(\ln(\Delta))}$. Spencer's result implies that the bound can not be improved if $\omega = \Omega(\Delta^{1/2})$. Considering this, we were motivated to answer the following question.

Question 1.5. *Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that, for every $c > 1$ and every graph G of maximum degree at most Δ with no clique of size greater than $\Delta^{1/f(c)}$, we have $\chi_\ell(G) \leq \Delta/c$?*

2 Results

Our first result is the following theorem.

Theorem 2.1. *If G is a graph of maximum degree at most Δ with no clique of size greater than ω, then*

$$\chi_\ell(G) = O\left(\Delta\sqrt{\frac{\ln(\omega)}{\ln(\Delta)}}\right).$$

Theorem 2.1 answers Question 1.5 in the affirmative with a function $f(c)$ that is quadratic in c; for large enough Δ, the function $f(c) = (72c)^2$ suffices. Determining the best possible function f that confirms Question 1.5 would be very interesting. As mentioned, Spencer [16] showed that $f(2) \geq 2$. Spencer's result actually provides a lower bound on $f(c)$ that is linear in c. Spencer [16] proved that the Ramsey number $R(c,\omega)$ is at least $\Omega\left((\omega/\ln(\omega))^{\frac{c+1}{2}}\right)$ as $\omega \to \infty$ for fixed $c \geq 3$. Therefore there exists a graph G on n vertices with no independent set of size c (and thus chromatic number at least $n/(c-1)$) and no clique of size ω where n is at least $\omega^{\frac{c+1}{2}-o(1)}$. Since the maximum degree of a graph is at most its number of vertices, it follows that $f(c) \geq c/2 + 1$ if $c \in \mathbb{N}$.

The bound of Spencer [16] was improved by Kim in [12] for $c = 3$ by a factor of $\ln\omega$ (matching the upper bound of Ajtai, Komlós, and Szemerédi [1] up to a constant factor), by Bohman in [3] for $c = 4$ by a factor of $\sqrt{\ln\omega}$, and by Bohman and Keevash in [4] for $c \geq 5$ by a factor of $\ln^{\frac{1}{c-2}}\omega$, but these improvements do not change the resulting lower bound on $f(c)$.

2.1 Local Versions

We actually prove a result much stronger than Theorem 2.1. One might wonder if the bounds on $|L(v)|$ supplied by Theorems 1.1, 1.2, 1.4, and 2.1 can be relaxed to depend on local parameters, such as the degree of the vertex v, or the size of a largest clique containing v, rather than the global parameters Δ and ω. To that end, for a vertex v, we let $\deg(v)$ denote the degree of v, $\omega(v)$ denote the size of a largest clique containing v, and $\chi(v)$ denote the chromatic number of the neighborhood of v.

We are interested in proving that a graph G is L-colorable whenever every vertex v satisfies $|L(v)| \geq f(v)$ where $f(v)$ depends on parameters such as $\deg(v)$ and $\omega(v)$. The archetypal example is the classical theorem of Erdős, Rubin, and Taylor [8] that a graph is *degree-choosable* (meaning L-colorable for any list-assignment L satisfying $|L(v)| = \deg(v)$ for every vertex v) unless every block is a clique or an odd cycle. We call such a Theorem a "local version." Our main result implies local versions of Theorems 1.1, 1.2, and 2.1 simultaneously, although we do not match the leading constant in Theorem 1.1.

In fact, we prove the theorem for *correspondence coloring*, a generalization of list-coloring introduced by Dvořák and Postle [7] in 2015, and also known as *DP-coloring*. For the sake of space, we do not define correspondence-coloring here; the theorem as stated below can also be read as if L is a list assignment.

Theorem 2.2. *For all sufficiently large Δ the following holds. Let G be a graph of maximum degree at most Δ with correspondence assignment (L, M). If for each $v \in V(G)$,*

$$|L(v)| \geq \deg(v) \cdot 72 \cdot \min\left\{\sqrt{\frac{\ln(\omega(v))}{\ln(\deg(v))}}, \frac{\omega(v)\ln(\ln(\deg(v)))}{\ln(\deg(v))}, \frac{\log_2(\chi(v)+1)}{\ln(\deg(v))}\right\},$$

and $\deg(v) \geq \ln^2(\Delta)$, then G is (L, M)-colorable.

Recently, Bernshteyn [2] proved that Theorems 1.1 and 1.2 hold for the *correspondence chromatic number*, which is always at least as large as the list-chromatic number. Our Theorem 2.2 implies that "local versions" of these theorems are true for correspondence coloring. It also implies a "local version" of a result of Johansson [10] on graphs that are *locally r-colorable*, meaning the neighborhood of every vertex is r-colorable. Of course, Theorem 2.2 also implies a "local version" of Theorem 2.1, which in turn implies Theorem 2.1.

Although we can not match the leading constant in Theorem 1.1 in our "local version," we can get the leading constant within a factor of $4\ln(2)$, as follows.

Theorem 2.3. *For every $\varepsilon > 0$, if Δ is sufficiently large and G is a graph of maximum degree at most Δ with correspondence assignment (L, M) such that for each $v \in V(G)$,*

$$|L(v)| \geq (4+\varepsilon)\frac{\deg(v)}{\log_2(\deg(v))}$$

and $\deg(v) \geq \ln^2(\Delta)$, then G is (L, M)-colorable.

2.2 A More General Theorem

As mentioned, Bernshteyn [2] proved that Theorems 1.1 and 1.2 hold for the correspondence chromatic number. Many aspects of Bernshteyn's proofs are similar to those of Molloy's [13]; however,

Bernshteyn's proof is much shorter and simpler. Molloy used a proof technique known as "entropy compression," which proves that a random algorithm terminates. Bernshteyn cleverly realized that the use of entropy compression in Molloy's proof can be replaced with the Lopsided Lovász Local Lemma, resulting in a substantial simplification of the proof.

Both proofs can be applied in the more general setting of graphs in which the average size of an independent set is somewhat large in comparison to the number of independent sets. We make this precise by extracting a more general theorem from their proofs, and we actually prove a "local version" of it, as follows.

For a graph H, let $\overline{\alpha}(H)$ and $i(H)$ denote the average size of an independent set and the number of independent sets in H respectively.

Theorem 2.4. *Let G be a graph of maximum degree at most Δ with correspondence-assignment (L, M), and $\varepsilon \in (0, 1/2)$. Let $\ell, t : V(G) \to \mathbb{N}$, and for each $v \in V(G)$, let $\alpha_{min}(v)$ be the minimum of $\overline{\alpha}(H)$ taken over all induced subgraphs $H \subseteq G[N(v)]$ such that $i(H) \geq t(v)$. If for each $v \in V(G)$,*

$$|L(v)| \geq \max\left\{(1+2\varepsilon)\frac{\deg(v)}{\alpha_{min}(v)}, \frac{2t(v)\ell(v)}{\varepsilon(\varepsilon - 2\varepsilon^2)}\right\},$$

and $\ell(v) \geq 18\ln(3\Delta)$, and $\binom{d(v)}{\ell(v)}/\ell(v)! < \Delta^{-3}/8$, then G is (L, M)-colorable.

We prove Theorems 2.2 and 2.3 using Theorem 2.4. We think that proving Theorem 2.4 separately makes the proof easier to follow, and we think that Theorem 2.4 may have other applications not listed here.

References

[1] M. Ajtai, J. Komlós, and E. Szemerédi. A note on Ramsey numbers. *J. Combin. Theory Ser. A*, 29(3):354–360, 1980.

[2] A. Bernshteyn. The Johansson–Molloy Theorem for DP-Coloring. *ArXiv e-prints*, August 2017.

[3] T. Bohman. The triangle-free process. *Adv. Math.*, 221(5):1653–1677, 2009.

[4] T. Bohman and P. Keevash. The early evolution of the H-free process. *Invent. Math.*, 181(2):291–336, 2010.

[5] M. Bonamy, T. Perrett, and L. Postle. Colouring graphs with sparse neighbourhoods: Bounds and applications. submitted.

[6] M. Delcourt and L. Postle. On the list coloring version of reed's conjecture. manuscript.

[7] Z. Dvořák and L. Postle. Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8. *J. Combin. Theory Ser. B*, 129:38–54, 2018.

[8] P. Erdős, A. L. Rubin, and H. Taylor. Choosability in graphs. In *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif., 1979)*, Congress. Numer., XXVI, pages 125–157. Utilitas Math., Winnipeg, Man., 1980.

[9] A. Johansson. Asymptotic choice number for triangle free graphs. Unpublished Manuscript, 1996.

[10] A. Johansson. The choice number of sparse graphs. Unpublished Manuscript, 1996.

[11] J. H. Kim. On Brooks' theorem for sparse graphs. *Combin. Probab. Comput.*, 4(2):97–132, 1995.

[12] J. H. Kim. The Ramsey number $R(3, t)$ has order of magnitude $t^2/\log t$. *Random Structures Algorithms*, 7(3):173–207, 1995.

[13] M. Molloy. The list chromatic number of graphs with small clique number. *ArXiv e-prints*, January 2017.

[14] S. Pettie and H-H. Su. Distributed coloring algorithms for triangle-free graphs. *Inform. and Comput.*, 243:263–280, 2015.

[15] B. Reed. ω, Δ, and χ. *J. Graph Theory*, 27(4):177–212, 1998.

[16] J. Spencer. Asymptotic lower bounds for Ramsey functions. *Discrete Math.*, 20(1):69–76, 1977/78.