Four bounds for the dichromatic number of a digraph

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Abstract

For a digraph D, we define the *dichromatic number*, $\chi_A(D)$, of D as the minimum k for which D admits a k-coloring of its vertices such that each color class induces an acyclic subdigraph in D. This number is related to de chromatic number of an undirected graph as follows: Let G be a graph and let D be digraph obtained from G by replacing each edge uv of G by a pair of symmetric arcs, (u, v) and (v, u); an acyclic coloring of D is a proper coloring of G, since each pair of adjacent vertices in G belong to a common 2-cycle in D and thus they must have different colors.

Determine the dichromatic number of a digraph is a difficult task. In fact, the authors in [2] proved that the problem of deciding if a given digraph D has dichromatic number equal to 2 is NP-complete, this is even true when D is a tournament. Therefore, a more reasonable task is to give bounds for these number. In this talk we will give four of these bounds.

In this talk $D = (V(D), A(D))$ will denote a loopless digraph. Two vertices u and v are adjacent if (u, v) or (v, u) is in $A(D)$. The *underlying graph* of D is the graph G obtained from D by replacing each arc (u, v) for the undirected edge uv. Given a subset S of $V(D)$ the *induced subdigraph* of D by S, denoted by $D\langle S\rangle$, is the subdigraph of D with vertex set S and $(u, v) \in A(D\langle S\rangle)$ if and only if $(u, v) \in A(D)$.

Here the walks, paths and cycles are always directed. The *girth* (resp. *circumference*) of D is the shortest (resp. longest) cycle. A *proper k-coloring* of a graph G is an assignment of k colors, 1, 2, ..., k, to the vertices of G such that each pair of adjacent vertices have different colors. A *proper k-coloring* of D is a proper k-coloring its underlying graph G. The *chromatic number* of a graph G (resp. digraph D), denoted by $\chi(G)$ (resp. $\chi(D)$) is the minimum k such that G (resp. D) admits a proper k-coloring. An *acyclic k-coloring* of D is a function $c : V(D) \to [k]$ such that $c^{-1}(i)$ induces an acyclic subdigraph in D for all $i \in [k]$, where $[k]$ denotes the set of positive integers $\{1, 2 \ldots, k\}$. The *dichromatic number* of D, denoted by $\chi_A(D)$, is the minimum k such that D admits an acyclic k-coloring.

For a strongly connected digraph D, we call a *DFS tree* to a subdigraph T with $V(D) = V(T)$ obtained by the DFS algorithm, which is an out-branching rooted at a vertex r, the *root*. The time a vertex $v \in V(T)$ is explored by the algorithm, $f(v)$, is the *DFS label* of v. Notice that each vertex has a unique DFS label and any two different vertices have different DFS labels. * Whenever there is a uv-path in T, v is a *descendant* of u and u is an *ancestor* of v. Recall that an arc $(u, v) \in A(D)$ is: (i) a *tree arc* if $(u, v) \in A(T)$, (ii) a *backward arc* if u is a descendant of v, (iii) a *forward arc* if u is an ancestor of v and (iv) a *cross arc* if u is neither an ancestor nor a descendant of v. For further details we refer the reader to [1] pages 26-29.

It is clear that if D is an acyclic digraph then $\chi_A(D) = 1$ and if D contains a cycle then $\chi_A(D) \geq 2$, therefore we look for upper bounds for $\chi_A(D)$.

In [5] Neumann-Lara, proved for a digraph D that the dichromatic number of D, $\chi_A(D)$, equals the maximum of the dichromatic numbers of the strong components of D. Thanks to this, we can focus our attention only on strongly connected digraphs.

In [6] Tuza used a depth first search (DFS) tree of a graph to study its proper colorings, now we will use the directed version of a DFS tree to study the dichromatic number of a digraph.

Henceforth, D will be a strongly connected digraph and T will be a DFS tree of D rooted at a vertex r and V will denote $V(D) = V(T)$.

Remark 1. It follows from the definitions that in D there are four types of arcs: tree arcs, forward arcs, backward arcs and cross arcs. If (u, v) is: (i) a tree arc then $f(u) < f(v)$, (ii) a forward arc then $f(u) < f(v)$, (iii) a backward arc then $f(u) > f(v)$ and (iv) a cross arc then $f(u) > f(v)$, which means that each cross arc goes from a branch explored later to a branch explored earlier (this assertion was proved in [4] page 524).

Remark 2. If $f(u) < f(v)$ and v is an out-neighbor of u in D, then there is a uv-path in T. Even more, if $f(u) < f(v)$ and v is a descendant of u, then for each w such that $f(u) < f(w) < f(v)$ we have w is also a descendant of u. This is due to the definition of the DFS algorithm, since all vertices explored between the time a vertex u is visited for the first time, namely $f(u)$, and before u is completely processed are descendants of u ([4], page 524).

Remark 3. Whenever there is a uv-path in T, this is unique in T, u must be an ancestor of v in T and $f(u) < f(v)$; moreover, every walk in T is a path.

Let D be a strongly connected digraph, T be a DFS tree of D rooted at r and t be the length of a longest path in T. As a consequence of the three Remarks, the *i-th generation* of T defined as the set $V_i = \{u \in V \mid$ the length of the (unique) $ru-$path is $i\}$ is well defined for each i, $0 \leq i \leq t$, and these sets form a partition of $V(D)$ into acyclic subsets, *i.e.* $D\langle V_i\rangle$ is acyclic for each i, $0 \leq i \leq t$. Then, when we look for acyclic colorings of $V(D)$, we can color all vertices in V_i with the same color.

We address the problem of coloring the vertices of D in two different ways to give upper bounds for the dichromatic number of D, by means of a DFS tree: one of them is based on the lengths of cycles and paths in D giving colorings to the generations of T; and the other one is based on the behavior of the backward arcs of D.

Let's start with the results relating $\chi_A(D)$ with the lengths of cycles and paths. Consider the girth, circumference and the lengths of longest paths in D, we have two bounds:

Proposition 1. *Let D be a strongly connected digraph with small diameter and large out-degree. Set $l = \min_{u\in V}\{j \mid$ j is the length of a longest path starting at $u\}$ and let g be the girth of D. Then $\chi_A(D) \leq \left\lceil \frac{l+1}{g-1} \right\rceil$.*

Proposition 2. *Let D be a digraph with at least one cycle and let c and g be its circumference and girth, respectively. Then $\chi_A(D) \leq \left\lceil \frac{c-1}{g-1} \right\rceil + 1$.*

In [3], authors proved for two integers r and k with $k \geq 2$ and $k \geq r \geq 1$ that if a digraph D contains no cycle of length r modulo k, then $\chi_A(D) \leq k$ (Theorem 1). We will consider the case where every cycle has length r modulo k.

Theorem 1. *[3] Let r and k be two integers with $k \geq 2$ and $k \geq r \geq 1$. If a digraph D contains no cycle of length r* modulo *k, then $\chi_A(D) \leq k$.*

Observe that this result allows us to relate the dichromatic number with the girth and circumference in a different way as follows:

Corollary 1. *Let D be a digraph with girth g and circumference c with $g-1 \leq c-g+2$ ($g \leq \frac{c+3}{2}$). Then $\chi_A(D) \leq c - g + 2$.*

The bound proposed in corollary 1 is sharp. If we take a digraph D such that all its cycles have length 3, then $2 \leq \chi_A(D)$ as D has cycles and $\chi_A(D) \leq 3 - 3 + 2 = 2$ by Corollary 1. Thus, $\chi_A(D) = 2$ and the bound is reached.

The third bound is the result we promised for digraphs that have all cycles of the same length modulo k.

Theorem 2. *Let k and r be two integers such that $k \geq r \geq 0$ and let D be a strongly connected digraph such that every cycle in D has length congruent with r modulo k. If $r \neq 1$ then $\chi_A(D) \leq 3$.*

If we restrict the parity of $k/\gcd(r-1,k)$, the bound can be improved.

Corollary 2. *Let k and r be two integers such that $k \geq r \geq 0$ and let D be a strongly connected digraph such that every cycle in D has length congruent with r modulo k. If $r \neq 1$ and $\frac{k}{d}$ is even, where $d = \gcd(r-1,k)$, then $\chi_A(D) \leq 2$.*

Now, let's forget the lengths of cycles and paths and recall backward arcs.

We noticed that, as a consequence of the DFS algorithm, every cycle in D have at least one backward arc. Remember that our goal is to color the vertices of D in such a way that no cycle is monochromatic, so it is sufficient to give different colors to the ends of each backward arc.

Consider a digraph D and a DFS tree T of D. Given a connected subdigraph H of T define the undirected graph G_H with vertex set $V(G_H) = V(H)$ and $uv \in E(G_H)$ whenever there is a backward arc between u and v in D. We will call G_H the *underlying backward graph* of H relative to T. Notice that G_H is always a subgraph of the underlying graph G of D, and it is a proper subgraph whenever there is a tree arc (u,v) such that $(v,u) \notin A(D)$ or whenever there is a cross arc in D. The fourth bound for $\chi_A(D)$ is in terms of $\chi(G_T)$.

Lemma 1. *Let D be a strongly connected digraph and T a DFS tree rooted at r. Let H be a connected subdigraph of T and G_H its underlying backward graph relative to T. Then $\chi_A(D\langle V(H)\rangle) \leq \chi(G_H)$.*

Observe that T is itself a connected subdigraph of T and thus $\chi_A(D\langle V(T)\rangle) \leq \chi(G_T)$. As $V(T) = V(D)$, we have $D = D\langle V(T)\rangle$, hence $\chi_A(D) \leq \chi(G_T)$.

There are many results on the chromatic number of a graph, to be able to use them, we must ask: what can we say about the structure of G_T?

We started with these four upper bounds for the dichromatic number of a digraph, but there is still work to be done using the DFS tree tool.

References

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