An Erdős-Gallai type theorem for vertex colored graphs

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Abstract

While investigating odd-cycle free hypergraphs, Győri and Lemons introduced a colored version of the classical theorem of Erdős and Gallai on P_k-free graphs. They proved that any graph G with a proper vertex coloring and no path of length $2k+1$ with endpoints of different colors has at most $2kn$ edges. We show that Erdős and Gallai's original sharp upper bound of kn holds for their problem as well. We also introduce a version of this problem for trees and present a generalization of the Erdős-Sós conjecture.

We denote by P_ℓ the path of length ℓ. For a graph G, we denote by $E(G)$ and $V(G)$ the edge and vertex set of G, respectively. We begin by recalling the theorems of Erdős and Gallai about graphs without long paths and cycles.

Theorem 1 (Erdős-Gallai [2])**.** *Let ℓ be a positive integer and G an n-vertex graph with no P_ℓ, then*

$$|E(G)| \leq \frac{\ell-1}{2}n,$$

and equality holds if and only if ℓ divides n and G is the disjoint union of $\frac{n}{\ell}$ cliques of size ℓ.

Theorem 2 (Erdős-Gallai [2])**.** *Let ℓ be a positive integer and G an n-vertex graph with no C_m for all $m \geq \ell$, then*

$$|E(G)| \leq \frac{(\ell-1)(n-1)}{2}.$$

and equality holds if and only if $\ell-2$ divides $n-1$ and G is a connected graph such that every block of G is a clique of size $\ell-1$.

In a more recent paper, Győri and Lemons [4] investigated the extremal number of hypergraphs avoiding so-called Berge-cycles. To this end, they introduced a generalization of the theorem of Erdős and Gallai about paths. By a proper vertex coloring of a graph G, we mean a coloring of the vertices of G such that no two adjacent vertices are the same color. Győri and Lemons proved the following.

Theorem 3 (Győri-Lemons [4])**.** *Let k be a positive integer and G be an n-vertex graph with a proper vertex coloring such that G contains no P_{2k+1} with endpoints of different colors, then*

$$|E(G)| \leq 2kn.$$

We show that the factor of 2 in Theorem 3 is not needed and, thus, recover the original upper bound from the Erdős-Gallai theorem. We also determine which graphs achieve this upper bound.

Theorem 4. *Let k be a positive integer and G be an n-vertex graph with a proper vertex coloring such that G contains no P_{2k+1} with endpoints of different colors, then*

$$|E(G)| \leq kn,$$

and equality holds if and only if $2k+1$ divides n and G is the union of $\frac{n}{2k+1}$ disjoint cliques of size $2k+1$.

Proof of Theorem 4. By induction on the number of vertices, we may assume that G is connected and has minimum degree $\delta(G) \geq k$.

If G is C_ℓ-free for all $\ell \geq 2k+1$, then by Theorem 2 we have $|E(G)| \leq \dfrac{(n-1)2k}{2} < kn$. Thus, assume there is a cycle of length at least $2k+1$, and let C be the smallest such cycle. Let ℓ be its length. Let the vertices of C be $v_0, v_1, v_2, \ldots, v_{\ell-1}, v_0$, consecutively. Addition and subtraction in subscripts will always be taken modulo ℓ. We say that an edge e is *outgoing* if it has one vertex in $V(C)$ and the other in $V(G) \setminus V(C)$. We say a vertex $v \in V(C)$ is outgoing if it is contained in an outgoing edge. We will consider cases based on the value of ℓ.

Note that G cannot contain a cycle of length $2k+2$ since any two consecutive vertices of such cycle would also be the endpoints of a path of length $2k+1$.

Case 1. Suppose $\ell \geq 2k+4$. Since we have chosen ℓ to be the length of the smallest C_ℓ with $\ell \geq 2k+1$, we have v_0 cannot be adjacent to any of $v_2, v_3, \ldots, v_{\ell-2k}$ nor any of $v_{2k}, v_{2k+1}, \ldots, v_{\ell-2}$, for otherwise we would have a shorter cycle of length at least $2k+1$. Also note that v_0 is adjacent to v_1 and $v_{\ell-1}$.

Observe that v_0 cannot have two consecutive neighbors in the ℓ-cycle. Indeed, if v_i and v_{i+1} are neighbors of v_0, then we have the following $(2k+1)$-paths starting at v_1: $v_1, v_2, \ldots, v_{2k+1}, v_{2k+2}$ and $v_1, v_2, \ldots, v_i, v_0, v_{i+1}, v_{i+2}, \ldots, v_{2k}, v_{2k+1}$. Thus, v_{2k+1} and v_{2k+2} would have to be colored the same, but this is impossible since they are neighbors.

If v_0 has a neighbor outside of C, say u_0, then we have two paths of length $2k+1$: $u_0, v_0, v_1, \ldots, v_{2k}$ and $v_{2k}, v_{2k-1}, \ldots, v_0, v_{\ell-1}$. It follows that u_0 and $v_{\ell-1}$ have the same color. Similarly, u_0 and v_1 have the same color. Thus, $v_{\ell-1}$ and v_1 also have the same color, and similarly, for every i such that v_i is outgoing, we can conclude v_{i-1} and v_{i+1} have the same color.

If $\ell = 2k+4$ and there is an outgoing vertex, say v_0, then v_1 and v_{2k+3} have the same color (from the previous paragraph), a contradiction since v_1 and v_{2k+2} also have the same color (they are endpoints of a length $2k+1$ path along the cycle C). If there is no outgoing vertex in $V(C)$, then C uses all vertices of the graph. Since no vertex of the cycle has two consecutive neighbors, it follows that each degree is bounded by $2 + \lceil \frac{2k-5}{2} \rceil \leq k$ and so the number of edges is at most $\frac{(2k+4)k}{2} = \frac{nk}{2} < nk$.

If $\ell \geq 2k+5$, we will show that v_0 has an outgoing edge from the ℓ-cycle C. Suppose not, then since v_0 does not have consecutive neighbors, it follows that v_0 has at most $2 + \left\lceil \dfrac{2k - (\ell - 2k + 1)}{2} \right\rceil \leq k-1$ neighbors, a contradiction. Thus, v_0 and similarly every other v_i has an outgoing neighbor, and it follows that for every i, the vertices v_i and v_{i+2} have the same color. Hence v_0 and v_{2k} have the same color, contradicting that v_0 and v_{2k+1} have the same color, since they are endpoints of a P_{2k+1}.

Case 2. Suppose $\ell = 2k+3$. For all $0 \leq i \leq \ell - 1$, $v_{i+2}, v_{i+1}, \ldots, v_{\ell-1}, v_0, \ldots, v_i$ is a path of length $2\ell + 1$, and so v_i and v_{i+2} have the same color. Thus, v_0 and v_{2k+2} have the same color, but they are adjacent, contradiction.

Case 3. Finally, suppose $\ell = 2k+1$. If no edge is outgoing, then we are done, since by connectivity the total number of edges in the graph is at most $\binom{2k+1}{2} = kn$. If indeed the total number of edges is kn, then G is a clique. This is the only case when equality holds. From here on, we will assume there is an outgoing edge.

Observe that if u is not a vertex of C, then u cannot have two consecutive neighbors in C, for otherwise we would have a cycle of length $2k+2$, and we are done. Moreover, u cannot be connected to v_i and v_{i+3}, since there would be two paths of length $2k+1$ from u to v_{i+1} and v_{i+2}.

It follows that u can have at most $k-1$ neighbors in C and, thus, must have a neighbor outside C.

If there are some two consecutive non outgoing vertices in C, then we may take two such vertices v_i and v_{i+1}, for some index i, so that the next vertex v_{i+2} is outgoing. Suppose $\{v_{i+2}, u\}$ is an outgoing edge. By the previous observation, there is an edge $\{u, w\}$ where $w \notin C$. So we have a $2k+1$ length path from v_i to w, then v_{i+1} cannot have two consecutive neighbors from C, since that would also imply that there is also $2k+1$ length path from w to v_{i-1}, similarly v_i cannot have two consecutive neighbors in C, hence v_i and v_{i+1} have degree at most k. By removing these two vertices, we remove $2k-1$ edges, and by the induction hypothesis the resulting graph has at most $k(n-2)$ edges. So $e(G) < kn$.

For every i, either v_{i+1} or v_{i+2} is an outgoing vertex. Hence the vertex v_i has either the same color as v_{i+2}, if v_{i+1} is an outgoing vertex, or the same color as v_{i+4}, if v_{i+2} is an outgoing vertex. Hence we obtain that v_0 has the same color as v_{2k} or v_1, contradiction. □

Recall that the extremal number $\mathrm{ex}(n, H)$ of a graph H is defined to be the largest number of edges an n-vertex graph may have if it does not contain H as a subgraph. We introduce a variation of the extremal function $\mathrm{ex}(n, T)$ in the case of trees. Let $\mathrm{ex}^c(n, T)$ denote the maximum number of edges possible in an n-vertex graph G with a proper vertex coloring, such that in every copy of T in G the leaves of T are all the same color. In the following we determine the value of $\mathrm{ex}^c(n, T)$ for every tree T such that every pair of leaves is at even distance.

Theorem 5. *Let ℓ be a positive integer and T a tree with k edges such that in the proper vertex 2-coloring of T all leaves are the same color, then $\mathrm{ex}^c(n, T) = \left\lfloor \frac{n^2}{4} \right\rfloor$, provided n is sufficiently large.*

Proof. The fact that all leaves are colored the same by a 2-coloring implies that all paths between a pair of leaves have even length. We add an edge e to T connecting an arbitrary pair of leaves, and let G be the resulting graph. Since G has an odd cycle, its chromatic number is clearly 3, and the deletion of e yields a 2-chromatic graph. It follows from a theorem of Simonovits [6] that if n is sufficiently large, the extremal number of G is precisely $\mathrm{ex}(n, G) = \left\lfloor \frac{n^2}{4} \right\rfloor$. Thus, in any n-vertex graph with more than $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges we have a copy T with two adjacent leaves, and so in any proper coloring of this graph we have a copy of T with leaves of at least 2 colors. It follows that $\mathrm{ex}^c(n, T) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$, and this bound is realized by the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. □

Remark 1. *The paths of even length P_{2k} are a special case of Theorem 5. Here better bounds on n are known to exist. For example, the result of Füredi [3] on the extremal number of odd cycles implies that $n \geq 4k$ is sufficient.*

For the family of trees without this property we have the following bound.

Theorem 6. *Let k be a positive integer and T be a tree with k edges such that in the (unique) proper vertex 2-coloring of T all leaves are not the same color, then $\mathrm{ex}^c(n, T) \leq (k-1)n$.*

Proof. There is a path of odd length in T with endpoints which are leaves. Let G be an n-vertex graph with more than $(k-1)n$ edges with a proper vertex coloring. We may find a subgraph G' of G with average degree at least that of G and minimum degree greater than $k-1$. The proper coloring of G induces a proper coloring of G' and so applying Theorem 3 we may find a copy of $P_{2\ell+1}$ in G' with endpoints of distinct colors. We may now build up the rest of the tree in a greedy fashion as every degree in G' is at least k and T has $k+1$ vertices. Thus, we have found a copy of T in the graph G with leaves of at least two colors. □

Erdős and Sós made the following famous conjecture about the extremal number of trees.

Conjecture 1 (Erdős-Sós [1])**.** *Let k be a positive integer and T be a tree with k edges, then* $\mathrm{ex}(n,T) \leq \frac{(k-1)n}{2}$.

Similarly as Conjecture 1, we believe that a strengthening of Theorem 6 should hold for trees whose 2-coloring yields two leaves of different colors.

Conjecture 2. *Let k be a positive integer and T be a tree with k edges such that in the proper vertex 2-coloring of T all leaves are not the same color, then* $\mathrm{ex}^c(n,T) \leq \frac{(k-1)n}{2}$.

Conjecture 2 holds for paths of even length by Theorem 4. We take another step towards this conjecture by proving it in the case of double stars.

Theorem 7. *For positive integers a and b, let $S_{a,b}$ denote the tree on $a+b+2$ vertices consisting of an edge $\{u,v\}$ where $|N(u)\setminus v| = a$, $|N(v)\setminus u| = b$ and $N(u)\cap N(v) = \varnothing$.*
We have $\mathrm{ex}^c(n,S_{a,b}) \leq \frac{a+b}{2}n$.

Proof. Let G be a vertex colored graph with $|E(G)| > |V(G)|\frac{a+b}{2}$. Without loss of generality, suppose $a \leq b$. We may assume by induction that $\delta(G) > \frac{a+b}{2} \geq a$. Since $\mathrm{ex}(m, S_{a,b}) = m\frac{a+b}{2}$ (see, for example [5]), it follows that G contains a copy of $S_{a,b}$. Suppose this copy is defined by the edge $\{u,v\}$ together with the disjoint sets $A \subseteq N(u)$, $B \subseteq N(v)$ with $|A| = a, |B| = b$. Now, if there is more than one color in $A \cup B$, then we are done. So suppose the color of all vertices in $A \cup B$ is the same. Hence $A \cup B$ is an independent set.

If u is not adjacent to some $w \in B$, since $|N(w)| \geq a+1$, we can pick $C \subseteq N(w)\setminus\{u,v\}$ of size a. So the edge $\{v,w\}$ together with the sets $B' = (B\cup\{u\})\setminus\{w\}$ and C define a $S_{a,b}$ where the colors of all vertices in C are different from the colors of $B'\setminus\{u\}$.

If u is adjacent to all $w \in B$, then fix $x \in B$. Since $|N(x)| \geq a+1$, we can pick $C \subseteq N(x)\setminus\{u\}$ of size a. Let $y \in A$ and define $B' = (B\cup\{y\})\setminus\{x\}$. Observe that $B' \subseteq N(u)$, and the edge $\{u,x\}$ together with the sets B' and C defines a $S_{a,b}$, where again the color of the vertices in C is different from the color of vertices in B'. □

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