# A characterization of graphs with Dilworth number at most 3

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**Abstract**

The forbidden induced subgraph characterization for the classes of graphs with Dilworth number at most 1 and 2 are known. We give the forbidden induced subgraph characterization for the class of graphs with Dilworth number at most 3. We conjecture that for any $k \in \mathbb{N}$, every forbidden induced subgraph for the class of graphs with Dilworth number at most $k$ has at most $3k+1$ vertices, and verify this for $k \leq 7$.

All graphs in this article are finite, simple and undirected. For a graph $G$, we let $\overline{G}$ be its complement. The disjoint union of two graphs $G$ and $H$ will be denoted by $G + H$. For $n \geq 1$, the disjoint union of $n$ copies of $G$ is written $nG$. For a vertex $v$, $N(v)$ will denote the set of neighbors of $v$ and $N[v] := \{v\} \cup N(v)$. Let $M(v)$ be the set of non-neighbors of $v$. If $U$ is a set of vertices of $G$, then $G[U]$ denotes the subgraph of $G$ induced by $U$. A class $\mathcal{G}$ of graphs is said to be *hereditary* if for every graph $G \in \mathcal{G}$ and every induced subgraph $H$ of $G$, we have $H \in \mathcal{G}$. A set $U$ of vertices in a graph $G$ is said to be *Dilworth* if for any distinct $u, v \in U$, $N(u) \setminus N[v] \neq \emptyset$ and $N(v) \setminus N[u] \neq \emptyset$. Given $u, v \in U$, a pair of vertices $\{u', v'\}$ with the property that $u' \in N(u) \setminus N[v]$ and $v' \in N(v) \setminus N[u]$, is called a *certifying pair* for $\{u, v\}$. Hence $U \subset V(G)$ is Dilworth if and only if there is a certifying pair for each pair $u, v \in U$. The *Dilworth number* of a graph $G$, denoted $D(G)$, is the size of a largest Dilworth set. For more information, see Chapter X in [10].

The following three fundamental lemmas describe the behavior of the Dilworth number with respect to induced subgraphs, complementation, and disjoint union.

**Lemma 1** (Monotonicity)**.** *Let $H$ be obtained from $G$ by deleting a vertex $v$. Then*

$$D(H) \leq D(G) \leq 2D(H) + 1.$$

**Lemma 2** (Invariance under complementation)**.** $D(\overline{G}) = D(G)$.

**Lemma 3** (Additivity)**.** *The Dilworth number of a graph is the sum of the Dilworth number of its non-trivial components.*

By virtue of monotonicity and invariance under complementation, the class of graphs with Dilworth number at most $k$ is a self-complementary hereditary graph class. What can we say about the forbidden induced subgraphs for such a class?

It is well known that graphs with Dilworth number 1 are precisely the threshold graphs (see [10]). A theorem of Chvátal and Hammer ([4]) says that a graph is threshold if and only if it contains none of $2K_2, C_4, P_4$ as an induced subgraph. Graphs with Dilworth number at most 2 have a nice threshold characterization (see [2]) and coincide with what are called $TS$-graphs. Benzaken et al. ([2]) also prove a forbidden induced subgraph characterization for the class: there are 21 such graphs. Hoang et al. ([8]) show that every graph with Dilworth number at most 3 contains either a simplicial or co-simplicial vertex. (A vertex in a graph is simplicial if its neighborhood is

a clique, and it is co-simplicial if it is simplicial in the complement.) Payan ([12]) shows that every graph with Dilworth number at most 4 is perfect. We further note that the Dilworth number of a graph can be computed in time $\mathcal{O}(n^3)$ (see for instance [10]). Nara [11] gives a forbidden induced subgraph characterization for split graphs with Dilworth number at most 3. In this paper we give the forbidden induced subgraph characterization for the class of graphs with Dilworth number at most 3. The following is our main result.

**Theorem 1.** *A graph has Dilworth number at most* 3 *if and only if it contains none of the* 499 *graphs shown in the figures in [9] as an induced subgraph.*

We show below that, for any $k \in \mathbb{N}$, the set of forbidden induced subgraphs for the class of graphs with Dilworth number at most $k$ is finite. For $k \geq 1$, let us denote the class of graphs with Dilworth number at most $k$ by $\mathcal{G}_k$. Let $Forb(\mathcal{G}_k)$ denote the set of forbidden induced subgraphs for the class of graphs with Dilworth number at most $k$.

**Theorem 2.** *Let $k \in \mathbb{N}$. If $G \in Forb(\mathcal{G}_k)$, then $G$ has at most $(k+1)^2$ vertices. In particular, the set $Forb(\mathcal{G}_k)$ is finite.*

*Proof.* Let $G \in Forb(\mathcal{G}_k)$. Since $D(G) > k$, there is a Dilworth set $W$ in $G$ with $|W| = k+1$. For any two distinct vertices $u, v$ in $W$, there is a certifying pair $\{a, b\}$ for $\{u, v\}$. (There might be several certifying pairs, just choose one.) Let $C$ be the union of all such certifying pairs. Then $|C| \leq 2\binom{|W|}{2}$. The set $W$ is still a Dilworth set in the subgraph induced by $C \cup W$. Thus $D(G[C \cup W]) > k$. Since every proper induced subgraph of $G$ has Dilworth number at most $k$, $|V(G)| = |C \cup W| \leq |W| + 2\binom{|W|}{2} = |W|^2 \leq (k+1)^2$. □

A vertex in a graph is *pendant* if it has degree 1. A vertex is *dominating* if it is adjacent to all the other vertices. A vertex is *isolated* if its degree is 0. The *domino* is the graph obtained from $C_6$ by adding an edge between two antipodal points. Two vertices $v, w$ are called *twins* if $N(v) \setminus \{w\} = N(w) \setminus \{v\}$. Being twins defines an equivalence relation on the vertex set. The partition of the vertex set in equivalence classes is called the *twin partition*. The *size* of the twin partition is the number of equivalence classes. A *preorder* on a set $S$ is a reflexive transitive relation on $S$. Let $G$ be a graph. The *vicinal preorder* of $G$ is the preorder on $V(G)$, where $u \leq v$ if $N(u) \subseteq N[v]$. Thus the Dilworth number of $G$ is the size of a largest antichain in the vicinal preorder of $G$. This explains why this number is named after R. P. Dilworth who proved his famous decomposition theorem [5] which states that the size of a largest antichain in a finite partially ordered set $P$ is equal to the smallest number of chains whose union is $P$.

We characterize graphs with Dilworth number at most 3 by giving the list of forbidden induced subgraphs. First, we begin with a series of lemmas.

**Lemma 4.** *If $v$ is a dominating or isolated vertex in a non-trivial graph $G$, then $D(G) = D(G \setminus v)$. In particular, graphs in* $\mathrm{Forb}(\mathcal{G}_3)$ *do not have any isolated or dominating vertices.*

**Lemma 5.** *A 3-vertex graph $G$ has $D(G) = 1$. A 4-vertex graph $G$ has $D(G) \leq 2$.*

**Proposition 3.** *The graph $C_5$ is the only 5-vertex graph in* $\mathrm{Forb}(\mathcal{G}_3)$.

We need some lemmas before we can determine the graphs $G \in \mathrm{Forb}(\mathcal{G}_3)$ with $|V(G)| = 6$.

**Lemma 6.** *There are* 23 *disconnected graphs in* $\mathrm{Forb}(\mathcal{G}_3)$. *They are obtained by taking the disjoint union of $K_2$ with a graph in* $\mathrm{Forb}(\mathcal{G}_2)$ *(other than $C_5$) and $C_4 + P_4, C_4 + C_4, P_4 + P_4$.*

**Lemma 7.** *Let $G$ be a connected graph with at least 3 vertices. If $G$ has $p$ pendant vertices, then $D(G) \leq |V(G)| - p$.*

**Lemma 8.** *Let $G$ be a graph and let $t$ be the size of the twin partition of $G$. Then $D(G) \leq t$.*

**Proposition 4.** *The graphs $P_6$, $C_6$, the domino, and their complements are the only 6-vertex graphs in* $\mathrm{Forb}(\mathcal{G}_3)$.

Write $D(k)$ for the maximum number of vertices in a graph in $Forb(\mathcal{G}_k)$.

Let $D = (V, A)$ be a directed graph, with $n := |V|$. A cycle $C$ in $D$ is *wrong* if all arrows in $C$ are directed in one direction, except for exactly one arrow. If $D$ contains no wrong cycles, then in particular $D$ has no loops and no parallel arcs.

**Theorem 5.** *If $D$ has no wrong cycles, then*

$$|A| \leq \ max\ \{2n - 2, \frac{n^2}{4}\}. \tag{1}$$

**Corollary 6.** *We have*

$$D(k) \leq \ max\ \{3k + 1, \frac{(k+1)^2}{4} + k + 1\} \tag{2}$$

*Proof.* Let $G = (V, E) \in \mathrm{Forb}(\mathcal{G}_k)$ with $|V|$ maximal, so $|V| = D(k)$. Let $U \subseteq V$ be a Dilworth set of size $|U| = k + 1$ and write $W := V \setminus U$. For every vertex $x \in W$, there is at least one pair of vertices $a_x, b_x \in U$ with the property that $a_x \in N(x), b_x \notin N(x)$ and such that all other vertices $y \in W$ with $y \neq x$ have either $a_x \notin N(y)$ or $b_x \in N(y)$. (So $x$ is necessary for distinguishing $a_x$ and $b_x$). If this were not the case, one could remove $x$ from $G$ without reducing the Dilworth number, which is in contradiction with the fact that $G \in \mathrm{Forb}(\mathcal{G}_k)$.

Consider a digraph $D = (U, A)$ with vertex set $U$ and arc set $A$ obtained as follows. For each vertex $x \in W$ choose one pair $a_x, b_x \in U$ with the above property and we let $(a_x, b_x) \in A$. So $|A| = |W|$. We claim that

$$D \textit{ contains no wrong cycles.} \tag{3}$$

Suppose otherwise. Let $C = (v_0, v_1, \ldots, v_k, v_0)$ be a wrong cycle (here $k > 0$, as it is clear that $D$ does not contain loops), with arrows $a_i := (v_i, v_{i+1})$ for $i = 0, \ldots, k-1$ and one 'wrong' arrow $w := (v_0, v_k)$. Arrow $a_i$ corresponds to a vertex $s_i$ in $W$ with neighbor $v_i$ and non-neighbor $v_{i+1}$, and furthermore $s_i$ is the only vertex in $W$ that simultaneously has $v_i$ as a neighbor and $v_{i+1}$ as a non-neighbor. Arrow $w$ corresponds to a vertex $t$ in $W$ with neighbor $v_0$ and nonneighbor $v_k$. We examine the neighbors of $t$. As $v_0 \in N(s_0)$, $v_1 \notin N(s_0)$ and $v_0 \in N(t)$, we have $v_1 \in N(t)$. By the same argument (where we now compare vertex $s_1$ with $t$), also $v_2$ is a neighbor of $t$. Continuing in this way, we find that $v_1, \ldots, v_k$ are neighbors of $t$. This is a contradiction, as $v_k$ is not a neighbor of $t$. So (3) holds.

By (3) and (1) we conclude that

$$|W| = |A| \leq \ \max\ \{2(k+1) - 2, \frac{(k+1)^2}{4}\}, \tag{4}$$

which implies (2), since $D(k) = |V| = |W| + |U| = |W| + k + 1$. □

Finding the graphs in $Forb(\mathcal{G}_3)$ on $7, 8, 9$ and 10 vertices by the same means as in the proof of Proposition 4 seems a tedious job. Therefore, we have used a simple computer program to determine these graphs. On $7, 8, 9$ and 10 vertices there are $19, 170, 236$ and 48 graphs in $Forb(\mathcal{G}_3)$, respectively. By Corollary 6 every graph in $Forb(\mathcal{G}_3)$ has at most 10 vertices. Lemmas 3 and 4 give the complete list of forbidden graphs up to 6 vertices. In total there are 499 graphs. These are shown in the various diagrams of [9].

**Proposition 7.** $D(k) \geq 3k + 1$.

*Proof.* Let $G$ be the graph with vertex set $X \cup Y \cup Z$, where $X = \{x_1, \cdots x_{k+1}\}, Y = \{y_1, \cdots y_k\}$ and $Z = \{z_1, \cdots z_k\}$. The vertices in $X$ form a clique and $Y \cup Z$ is a stable set. For $1 \leq i \leq k$, the vertex $y_i$ has only one neighbor, $x_i$. The vertex $z_i$ (which has no neighbor in $Y$) has only one non-neighbor in $X$, namely $x_i$. It is easy to check that $G \in \text{Forb}(\mathcal{G}_k)$. Thus $D(k) \geq 3k + 1$. □

Proposition 7 and Corollary 6 imply that $D(k) = 3k + 1$, for $1 \leq k \leq 6$. For $k = 7$ we find $22 \leq D(7) \leq 24$. We prove (see [9]) that $D(7)$ equals the lower bound. We offer the reader the following conjecture.

**Conjecture 8.** $D(k) = 3k + 1$.

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