Uniform Orderings for Generalised Colouring Numbers and Graph Classes with Bounded Expansion [1]

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Abstract

The generalised colouring numbers were introduced by Kierstead and Yang as a generalisation of the usual colouring number, and have since found important theoretical and algorithmic applications. The definitions of these numbers involve a given distance and linear orderings of the vertices of the graphs. For different distances, the optimal orderings can be completely different. We show that it is possible to find a single uniform ordering that approximates (in a precisely prescribed way) all the generalised colouring numbers of that graph.

These uniform orderings also provide a new characterisation of graph classes of bounded expansion, where for every class in the graph we need only find one uniform ordering.

1 Introduction

The *colouring number* $\mathrm{col}(G)$ of a graph G is the minimum integer k such that there is a strict linear order $<_L$ of the vertices of G for which each vertex v has at most $k-1$ neighbours u with $u <_L v$. It is well-known that for any graph G, the chromatic number $\chi(G)$ satisfies $\chi(G) \leq \mathrm{col}(G)$.

Several generalisations of the colouring number have been studied in the literature. These include the *arrangeability* [1], the *admissibility* [9], and the *rank* [8]. But maybe the most natural generalisation of the colouring number are the two series scol_r and wcol_r of *generalised colouring numbers* introduced by Kierstead and Yang [10] in the context of colouring games and marking games on graphs. As proved by Zhu [17], these invariants are strongly related to low tree-depth decompositions [12], and can be used to characterise bounded expansion classes of graphs (introduced in [13], see also below) and nowhere dense classes of graphs (introduced in [14]).

The *strong r-colouring number* $\mathrm{scol}_r(G)$ of a graph G is the minimum integer k such that there is a linear order $<_L$ of the vertices for which each vertex v can reach at most $k-1$ other vertices smaller than v (in the order $<_L$) with a path of length at most r, all internal vertices of which are greater than v. For the *weak r-colouring number* $\mathrm{wcol}_r(G)$, we do not require that the internal vertices are greater than v, but only that they are greater than the final vertex of the path. (Formal definitions will be given below.) As noticed already in [10], the two types of generalised colouring numbers are related by the inequalities

$$\mathrm{scol}_r(G) \leq \mathrm{wcol}_r(G) \leq (\mathrm{scol}_r(G))^r. \tag{1}$$

Note that this means that if one of the generalised colouring numbers is bounded for a class of graphs (for some r), then so is the other one.

If we allow paths of any length (but still have similar restrictions on the position of the internal vertices), we get $\mathrm{scol}_\infty(G)$ and $\mathrm{wcol}_\infty(G)$.

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An interesting aspect of generalised colouring numbers is that they can also be seen as gradations between the colouring number $\mathrm{col}(G)$ and two important minor monotone invariants, namely the *tree-width* $\mathrm{tw}(G)$ and the *tree-depth* $\mathrm{td}(G)$ (which is the minimum height of a depth-first search tree for a supergraph of G [12]). More explicitly, for every graph G we have the following relations.

Proposition 1.

(a) $\mathrm{col}(G) = \mathrm{scol}_1(G) \leq \mathrm{scol}_2(G) \leq \ldots \leq \mathrm{scol}_\infty(G) = \mathrm{tw}(G) + 1$;

(b) $\mathrm{col}(G) = \mathrm{wcol}_1(G) \leq \mathrm{wcol}_2(G) \leq \ldots \leq \mathrm{wcol}_\infty(G) = \mathrm{td}(G)$.

The equality $\mathrm{scol}_\infty(G) = \mathrm{tw}(G) + 1$ was first proved in [4]. The equality $\mathrm{wcol}_\infty(G) = \mathrm{td}(G)$ is proved in [15, Lemma 6.5].

Shortly after Nešetřil and Ossona de Mendez [13] introduced the notion of *classes with bounded expansion*, Zhu [17] provided a characterising of these classes in terms of generalised colouring numbers. We will use this characterisation as a definition.

Definition 1.

A class of graphs $\mathcal{G}$ has *bounded expansion* if and only if there exists a function $c : \mathbb{N} \to \mathbb{R}$ such that $\mathrm{scol}_r(G) \leq c(r)$ for all r and all $G \in \mathcal{G}$.

Note that by the inequalities in (1) we equally well could have defined bounded expansion in terms of the weak colouring number.

The concept of a class of graphs with bounded expansion generalises that of classes of graph with bounded tree-width, minor-closed classes, etc. See the book of Nešetřil and Ossona de Mendez [15] for a wealth of information about the properties of these graph classes.

The generalised colouring numbers are an important tool in the context of algorithmic sparse graphs theory; see again [15]. More recently they've played a key role in algorithmic results on model-checking for first-order logic on bounded expansion and nowhere dense graph classes [3, 5, 7].

2 Orderings and Generalised Colouring Numbers

We now give a more formal definition of the generalised colouring numbers. For a graph G, let $\Pi(G)$ be the set of all linear orderings of $V(G)$, and let $L \in \Pi(G)$. For readability, we write $u <_L v$ if u is smaller than v with respect to L, and $u \leqslant_L v$ if $u <_L v$ or $u = v$.

Let $u, v \in V(G)$. For a positive integer r we say that u is *weakly r-reachable* from v with respect to L, if $u \leqslant_L v$ and there exists a path P of length $\ell \leq r$ between u and v such that u is minimum among the vertices of P (with respect to L). Let $W_r[G, L, v]$ be the set of vertices that are weakly r-reachable from v with respect to L. Note that $v \in W_r[G, L, v]$.

Next, u is *strongly r-reachable* from v with respect to L, if $u \leqslant_L v$ and there is a path P of length $\ell \leq r$ connecting u and v such that all internal vertices w of P satisfy $v <_L w$. Let $S_r[G, L, v]$ be the set of vertices that are strongly r-reachable from v with respect to L. Again we have $v \in S_r[G, L, v]$.

For a graph G and linear ordering $L \in \Pi(G)$ we define

$$\mathrm{wcol}_r(G, L) = \max_{v \in V(G)} \big|W_r[G, L, v]\big| \qquad \text{and} \qquad \mathrm{scol}_r(G, L) = \max_{v \in V(G)} \big|S_r[G, L, v]\big|.$$

Now $\mathrm{wcol}_r(G)$ and $\mathrm{scol}_r(G)$ are defined as

$$\mathrm{wcol}_r(G) = \min_{L \in \Pi(G)} \mathrm{wcol}_r(G, L) \qquad \text{and} \qquad \mathrm{scol}_r(G) = \min_{L \in \Pi(G)} \mathrm{scol}_r(G, L).$$

An obvious question related to these definitions is if an ordering that is "good" for some r is also "good" for a different r'. In fact, this need not be the case, as the following examples show. For integers n, t, define $G_{n,t}$ as follows. Start with the complete graph K_n with vertices $v_1, \ldots, v_n$. First subdivide each edge; i.e. replace each edge v_iv_j, $i \neq j$, by a path $v_iu_{i,j}v_j$ using a new vertex u_{ij}. Next replace each v_i by t vertices $v_i^1, \ldots, v_i^t$ which are adjacent to the same vertices $u_{i,j}$, $j \in \{1, \ldots, n\} \setminus \{i\}$ as v_i was.

We define two orderings of $V(G_{n,t})$: L_1 is the ordering in which the vertices v_i^s come first and then the vertices $u_{i,j}$; while L_2 is the ordering in which the vertices $u_{i,j}$ come first and then the vertices v_i^s. (The internal ordering of the two groups of vertices can be arbitrary in both cases.) Then it is not so hard to show that for $1 \ll t \ll n$ we have:

$$\begin{aligned} \mathrm{scol}_1(G) &= \mathrm{scol}_1(G, L_1) = 2t + 1 \ll n = \mathrm{scol}_1(G, L_2) \qquad \text{and} \\ \mathrm{scol}_2(G) &= \mathrm{scol}_2(G, L_2) = n \ll tn = \mathrm{scol}_2(G, L_1). \end{aligned}$$

So the ordering L_1 that is "good" for $r = 1$ is very "bad" for $r' = 2$; while the reverse is true for the ordering L_2. This construction can be generalised to arbitrary values of r, r' by replacing the vertices $u_{i,j}$ by paths and the edges $v_iu_{i,j}$ by paths, where the appropriate lengths of these replacement paths depend on r and r'.

The observation above also has consequences for the many algorithms that for a graph class $\mathcal{G}$ with bounded expansion, for some r, use explicitly an ordering L which shows that $\mathrm{scol}_r(G) \leq c(r)$. It seems that for every r a different ordering is needed.

Our main results show that in fact this is not the case.

Theorem 2. [6]
For any graph G, there exits a linear ordering L^* of $V(G)$ such that for all $r \in \mathbb{N}$ we have

$$\mathrm{scol}_r(G, L^*) \leq (2^r + 1) \cdot \big(\mathrm{scol}_{2r}(G)\big)^{4r}.$$

Corollary 3. [6]
A graph class $\mathcal{G}$ has bounded expansion if and only if there exists a function $c' : \mathbb{N} \to \mathbb{R}$, such that for every graph $G \in \mathcal{G}$ there exists an ordering $L^*(G)$ of $V(G)$ such that $\mathrm{scol}_r(G, L^*(G)) \leq c'(r)$ for all r.

Corollary 3 answers in the positive a problem raised by Dvořák (see [16]).

The results above follow from a technical, more general, result that deals with different graphs on the same vertex set. A specific consequence of this more general result is the following.

Theorem 4. [6]
Let $G_1, \ldots, G_k$ be a collection of graphs, all on the same vertex set V, and let $r_1, \ldots, r_k \in \mathbb{N}$. Then there exists a linear ordering L^* of V such that for all $i = 1, \ldots, k$ we have

$$\mathrm{scol}_{r_i}(G_i, L^*) \leq (k + 1) \cdot \big(\mathrm{scol}_{2r_i}(G_i)\big)^{4r_i}.$$

The proof of the general result is based on ideas in [10, 11].

The proofs are constructive and in fact give a polynomial algorithm to construct the uniform ordering L^*, assuming that orderings L such that $\mathrm{scol}_{2r}(G) = \mathrm{scol}_{2r}(G, L)$ are given for the relevant r. Dvořák [2] gives a polynomial algorithm for approximating the weak colouring numbers. By the relations in (1), this algorithm can be used, for given G and r, to find in polynomial time an ordering L of $V(G)$ such that $\mathrm{scol}_r(G, L)$ is bounded by a function of $\mathrm{scol}_r(G)$. Combining this with our algorithm to find the uniform ordering gives polynomial algorithms to find the uniform orderings for the results above as well.

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