Graphs and 3-uniform linear hypergraphs with no 5-cycle

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Abstract

In this talk, we investigate graphs and 3-uniform linear hypergraphs not containing any copy of the cycle C_5 (in Berge sense).

We study the maximum number of triangles in C_5-free graphs, first studied by Bollobás and the speaker, improved later by Alon and Shikhelman and now by Ergemlidze, Methuku, Salia and by the speaker. The best present bounds are $\frac{1}{3\sqrt{3}}(1+o(1))n^{3/2} \leq ex(n, K_3, C_5) \leq \frac{1}{2\sqrt{2}}(1+o(1))n^{3/2}$. In C_{2k+1}-free graphs it was studied by Hao Li and the speaker, then it was improved by Füredi an Özhahya, but here the lower and upper bounds are much farther from each other.

The number of hyperedges in C_5-free 3-uniform hypergraphs first was studied by Bollobás and the speaker too, and it was improved by Füredi an Özhahya. For C_{2k+1}-free hypergraphs, the extremal number was estimated by Lemons and the speaker. Recently, we studied it in 3-uniform linear hypergraphs. Let $\mathbf{F}$ be a family of 3-uniform linear hypergraphs.The linear Turán number of $\mathbf{F}$ is the maximum possible number of edges in a 3-uniform linear hypergraph on n vertices which contains no member of $\mathbf{F}$ as a subhypergraph. We asymptotically determine this Turán number for C_5.

The talk is based on the papers [8] , [7].

1 C_5-free graphs

Motivated by a conjecture of Erdős [5] on the maximum possible number of pentagons in a triangle-free graph, Bollobás and Győri [2] initiated the study of the natural converse of this problem. Let $ex(n, K_3, C_5)$ denote the maximum possible number of triangles in a graph on n vertices without containing a cycle of length five as a subgraph. Bollobás and Győri [2] showed that

$$\frac{1}{3\sqrt{3}}(1+o(1))n^{3/2} \leq ex(n, K_3, C_5) \leq \frac{5}{4}(1+o(1))n^{3/2}. \tag{1}$$

Their lower bound comes from the following example: Take a C_4-free bipartite graph G_0 on $n/3+n/3$ vertices with about $(n/3)^{3/2}$ edges and double each vertex in one of the color classes and add an edge joining the old and the new copy to produce a graph G. Then, it is easy to check that G contains no C_5 and it has $(n/3)^{3/2}$ triangles.

Recently, Füredi and Özkahya [9] gave a simpler proof showing a slighly weaker upper bound of $\sqrt{3}n^{3/2} + O(n)$. Alon and Shikhelman [1] improved these results by showing that

$$ex(n, K_3, C_5) \leq \frac{\sqrt{3}}{2}(1+o(1))n^{3/2}. \tag{2}$$

In fact, in their nice paper, they investigate the more general function $ex(n, T, H)$ which stands for the maximum possible number of copies of T in a H-free graph on n vertices.

We improve the previous estimates (1) and (2) by showing the following.

Theorem 1. *(Ergemlidze, Győri, Methuku, Salia [8])*

$$ex(n, K_3, C_5) \leq \frac{1}{2\sqrt{2}}(1 + o(1))n^{3/2}.$$

The main idea is that we can select an edge from each triangle constituting a C_4-free subgraph (so the number of edges in the subgraph is the same as the number of triangles in the original graph) and then we apply the following well-known theorem of Erdős and Simonovits [6].

Theorem 2. *(Erdős, Simonovits [6]) The maximum possible number of edges in a graph on n vertices containing no C_4 or C_5 as a subgraph is at most $\frac{1}{2\sqrt{2}}(1 + o(1))n^{3/2}$.*

2 C_5-free hypergraphs

Let $\mathcal{F}$ be a family of 3-uniform linear hypergraphs. The *linear Turán number* of $\mathcal{F}$ is the maximum possible number of edges in a 3-uniform linear hypergraph on n vertices which contains no member of $\mathcal{F}$ as a subhypergraph.

In this paper we show that the linear Turán number of the five cycle C_5 (in Berge sense) is $\frac{1}{3\sqrt{3}}n^{3/2}$ asymptotically. We also show that the linear Turán number of the four cycle C_4 and $\{C_3, C_4\}$ are equal asymptotically, which is a strengthening of a theorem of Lazebnik and Verstraëte [**?**].

A hypergraph $H = (V, E)$ is a family E of distinct subsets of a finite set V. The members of E are called hyperedges and the elements of V are called vertices. It is called r-uniform is each member of E has size r. A hypergraph $H = (V, E)$ is called *linear* if every two hyperedges have at most one vertex in common. A hypergraph is $\mathcal{F}$-free if it doesn't contain any member of $\mathcal{F}$ as a subhypergraph. A 2-uniform hypergraph is simply called a graph.

Given a family of graphs $\mathcal{F}$, the *Turán number* of $\mathcal{F}$, denoted $ex(n, \mathcal{F})$, is the maximum number of edges in an $\mathcal{F}$-free graph on n vertices and the *bipartite Turán number* of $\mathcal{F}$, denoted $ex_{bip}(n, \mathcal{F})$ is the maximum number of edges in an $\mathcal{F}$-free bipartite graph on n vertices. Given a family of 3-uniform hypergraphs $\mathcal{F}$, let $ex_3(n, \mathcal{F})$ denote the maximum number of hyperedges of an $\mathcal{F}$-free 3-uniform hypergraph on n vertices and similarly, given a family of 3-uniform linear hypergraphs $\mathcal{F}$, the *linear Turán number* of $\mathcal{F}$, denoted $ex_3^{lin}(n, \mathcal{F})$, is the maximum number of hyperedges in an $\mathcal{F}$-free 3-uniform linear hypergraph on n vertices. When $\mathcal{F} = \{F\}$ then we simply write $ex_3^{lin}(n, F)$ instead of $ex_3^{lin}(n, \{F\})$.

A cycle C_k of length k is an alternating sequence of distinct vertices and distinct edges of the form $v_1, h_1, v_2, h_2, \ldots, v_k, h_k$ where $v_i, v_{i+1} \in h_i$ for each $i \in \{1, 2, \ldots, k-1\}$ and $v_k, v_1 \in h_k$. This definition of a hypergraph cycle is the classical definition due to Berge. For $k \geq 2$, Füredi and Özkahya [9] showed $ex_3^{lin}(n, C_{2k+1}) \leq 2kn^{1+1/k} + 9kn$. In fact it is shown in [12, 9] that $ex_3(n, C_{2k+1}) \leq O(n^{1+1/k})$. For the even case it is easy to show $ex_3^{lin}(n, C_{2k}) \leq ex(n, C_{2k}) = O(n^{1+1/k})$ by selecting a pair from each hyperedge of a C_{2k}-free 3-uniform linear hypergraph. A (Berge) path P_k of length k is an alternating sequence of distinct vertices and distinct edges of the

form $v_1, h_1, v_2, h_2, \ldots, v_{k-1}, h_{k-1}, v_k$ where $v_i, v_{i+1} \in h_i$ for each $i \in \{1, 2, \ldots, k-1\}$. Recently the notion of Berge cycles and Berge paths was generalized to arbitrary Berge graphs in [10] and the linear Turán number of Berge-$K_{2,t}$ was studied in [14] and [11]. Below we concentrate on the linear Turán numbers of C_3, C_4 and C_5.

Determining $ex_3^{lin}(n, C_3)$ is basically equivalent to the famous $(6, 3)$-problem which is a special case of a general problem of Brown, Erdős, and Sós [3]. This was settled by Ruzsa and Szemerédi in their classical paper [13], showing that $n^{2-\frac{c}{\sqrt{\log n}}} < ex_3^{lin}(n, C_3) = o(n^2)$ for some constant $c > 0$.

Only a handful of results are known about the asymptotic behaviour of Turán numbers for hypergraphs. We determine the asymptotics of $ex_3^{lin}(n, C_5)$ by giving a new construction, and a new proof of the upper bound which introduces some important ideas. We also determine the asymptotics of $ex_3^{lin}(n, C_4)$ and construct 3-uniform linear hypergraphs avoiding linear cycles of given odd length(s). Surprisingly, even though the C_5-free hypergraph that Bollobás and Győri constructed in order to establish their lower bound has the same size as the C_5-free hypergraph we constructed in order to obtain the lower bound in our Theorem 3 below, these two constructions are quite different. Their hypergraph is very far from being linear.

The following is our main result.

Theorem 3. *(Ergemlidze, Győri, Methuku [7])*

$$ex_3^{lin}(n, C_5) = \frac{1}{3\sqrt{3}} n^{3/2} + O\left(\sqrt{n}\right).$$

For the lower bound, $ex_3^{lin}(n, C_5) \geq n^{3/2}/3\sqrt{3}$ we give the following construction.

Construction of a C_5-free linear hypergraph H: For each $1 \leq t \leq \sqrt{n/3}$, let $L_t = \{l_1^t, l_2^t, \ldots, l_{\sqrt{n/3}}^t\}$ and $R_t = \{r_1^t, r_2^t, \ldots, r_{\sqrt{n/3}}^t\}$. Let $B = \{v_{i,j} \mid 1 \leq i, j \leq \sqrt{n/3}\}$. The vertex set of H is $V(H) := \bigcup_{i=1}^{\sqrt{n/3}} (L_i \cup R_i) \cup B$ and the edge set of H is $E(H) := \{v_{i,j} l_i^t r_j^t \mid v_{i,j} \in B \text{ and } 1 \leq t \leq \sqrt{n/3}\}$.

Clearly $|V(H)| = n$ and $|E(H)| = n^{3/2}/3\sqrt{3}$ and H is linear. It is easy to check that H is C_5-free.

We also show that the linear Turán number of the four cycle C_4 and $\{C_3, C_4\}$ are equal asymptotically, which is a strengthening of a theorem of Lazebnik and Verstraete. We establish a connection between the linear Turán number of the linear cycle of length $2k+1$ and the extremal number of edges in a graph of girth more than $2k-2$ too. Combining our result and a theorem of Collier-Cartaino, Graber and Jiang, we obtain that the linear Turán number of the linear cycle of length $2k+1$ is $\Theta(n^{1+1/k})$ for $k = 2, 3, 4, 6$.

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