The price of connectivity for domination

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Abstract

The price of connectivity for dominating set in a graph G is the ratio between the minimum sizes of a connected dominating set and a dominating set of G. It is always at most three, and Zverovich characterized the class of graphs such that this ratio equals one. In this paper, we prove a conjecture of Camby and Schaudt by characterizing the class of graphs with price of connectivity at most two.

A *dominating set* of a graph is a subset of vertices such that every vertex of the graph is at distance at most one of a vertex of the dominating set. In the following, we denote by $\gamma(G)$ the size of a minimum dominating set of G. Informally, a dominating set represents a subset of agents of a network which can communicate with any other agent directly. Determining the minimum size of a dominating set is a classical NP-hard problem (one of Karp's 21 problems). The problem remains intractable even in restricted classes of graphs [14, 15].

In practice, dominating sets are not always satisfying. Indeed, the vertices of a dominating set may have to communicate between themselves before making a decision. It motivates the introduction of a variant of dominating sets: connected dominating sets [13]. A connected dominating set is a dominating set which induces a connected subgraph. Every connected graph admits a connected dominating set: the whole set of vertices is a dominating set. Connected dominating sets have several applications, such as routing messages in ad-hoc networks [2] or designing fiber optics networks [7]. However, any dominating set of size k can be transformed into a connected dominating set of size $(3k-2)$, as first observed in [9]. Conversely, a connected dominating set is indeed a dominating set. So, denoting by $\gamma_c(G)$ the size of a minimum connected dominating set of G, we have:

$$\gamma(G) \leq \gamma_c(G) \leq 3 \cdot \gamma(G)$$

In other words, $\gamma_c(G)$ and $\gamma(G)$ are of the same order: the only unknown concerns the exact ratio between the two. The *price of connectivity for dominating set* of the graph G, or $PoC(G)$ for short, is the maximum over all induced subgraphs G' of G of the ratio $\gamma_c(G')/\gamma(G')$ (its definition is similar to that of the price of anarchy, a well-known invariant in game theory [11]). This notion was first formally introduced by Cardinal and Levy [6], in the context of vertex cover. It has then been generalized to dominating sets [3] and feedback vertex sets [1]. We briefly note that the area has been active in the past few years [4, 5, 10].

Price of connectivity for dominating sets. A natural question is the following: which constraints on the graph ensure that the gap between $\gamma(G)$ and $\gamma_c(G)$ is small? To restate the earlier remark, the price of connectivity for dominating set is at most 3 [9]. Moreover, Camby and Schaudt argued in [3] that this bound of 3 is asymptotically tight. This can be seen by considering a very long path.

In fact, paths (and their close relatives, cycles) appear naturally when we try to bound the price of connectivity. Zverovich proved [12] that the class of (P_5, C_5)-free graphs is exactly the class of graphs for which the price of connectivity for dominating set is 1. Camby and Schaudt [3] extended this result by characterizing the class of graphs for which the difference between dominating set and connected dominating set is at most one. On the other hand, they proved that the class of (P_9, C_9)-free graphs contain graphs whose price of connectivity is arbitrarily close to 3.

Camby and Schaudt proved that the class of (P_8, C_8)-free graphs has price of connectivity at most two and that this bound is already tight for (P_7, C_7)-free graphs. Moreover, they pointed out that the graph represented in Figure 1, which we refer to as H from now on, has price of connectivity more than 2. Note that this is also the case of P_9 and C_9. Determining the class of graphs for which the price of connectivity is at most 2 was still open, and they conjectured that they are precisely the graphs which do not contain P_9, C_9 nor H as an induced subgraph.

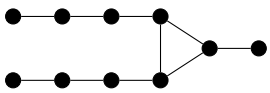


Figure 1: The graph H.

We prove here that the conjecture of Camby and Schaudt is satisfied, i.e. we characterize the class of graphs such that the price of connectivity for dominating set is at most 2.

Theorem 1. *Any (P_9, C_9, H)-free graph G satisfies $\gamma_c(G) \leq 2\gamma(G)$.*

We prove Theorem 1 by contradiction. Given a counter-example, we consider among the minimum dominating sets of G one with the fewest connected components. We then discuss how to reduce the number of connected components without increasing too much the number of vertices. Once the dominating set cannot be thus improved anymore, we define the notion of semi-relevant set as, informally, a set of vertices which would almost be worth adding to the dominating set. We discuss their properties, and derive a small connected dominating set, a contradiction.

Due to technical issues at the very end of the proof, we have to guarantee some notion of heredity on the connected components of the successive dominating sets. This accounts for the somewhat counter-intuitive definition of improvement.

Sketch of the proof of the main theorem

Let $G = (V, E)$ be a connected (P_9, C_9, H)-free graph. We assume for a contradiction that $\gamma_c(G) \geq 2\gamma(G) + 1$. Let D^0 be a minimum dominating set of G with the fewest connected components. Note that $|D^0| = \gamma(G)$ and $N[D^0] = V$. Let us denote by $D^0_1, \ldots, D^0_g$ the connected components of $G[D^0]$. Since $\gamma_c(G) \geq 2\gamma(G) + 1$, we have $g \geq 2$.

We consider how much D^0 can be improved with respect to the number of connected components. Let ℓ be the largest integer such that exists a dominating set D^ℓ of G with the following properties:

- $G[D^\ell]$ has $g - \ell$ connected components.
- $|D^\ell| \leq \gamma(G) + \ell$.
- There is an increasing function σ such that for every connected component D^ℓ_i of $G[D^\ell]$, $N[D^0_{\sigma(i-1)+1} \cup \ldots \cup D^0_{\sigma(i)}] \subseteq N[D^\ell_i]$, where $\sigma(0) = 0$ and $\sigma(g - \ell) = g$.

We emphasize that D^0 might not be a subset of D^ℓ.

Among the eligible such dominating sets, we take one of smallest size, and write for short $D = D^\ell$. Note that possibly $\ell = 0$. Again, since $\gamma_c(G) \geq 2\gamma(G) + 1$, we have $\ell \leq g - 2$. Let us denote by $D_1, \ldots, D_{g-\ell}$ the connected components of $G[D]$ and observe that for every $i \neq j$, no vertex has both a neighbor in D_i and D_j.

Observation 2. *For any set $T \subseteq \cup_i\{D_i\}$, the distance between T and $\cup_i\{D_i\} \setminus T$ is at most 3.*

We define the following useful notions. A set $S \subseteq V \setminus D$ is *semi-relevant* if $G[D \cup S]$ has at most $g - \ell - |S| + 1$ connected components. Note that by Observation 2, every D_i is incident to a semi-relevant set of size 2. Besides, one can prove that every semi-relevant set is connected. Given a semi-relevant set S, we define its *action* $A(S)$ as the set of connected components of $G[D]$ that it is adjacent to. Note that since $\gamma_c(G) \geq 2\gamma(G) + 1$, there is no semi-relevant set S with $A(S) = \cup_i D_i$. Moreover, there is a bijection between S and $A(S)$ where every component of $A(S)$ is adjacent to a unique vertex of S. Indeed, note that by maximality of ℓ, there is no vertex of S adjacent to two elements of $A(S)$. The statement follows by cardinality.

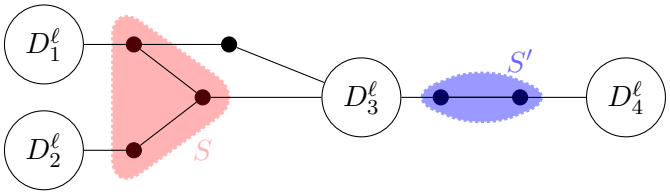


Figure 2: Two semi-relevant sets S and S'.

By abuse of notation, for every semi-relevant set S and for every $D_i \in A(S)$, we refer to the vertex of S adjacent to D_i as v_i^S. We say a semi-relevant set S is *maximal* if there is no semi-relevant set S' with $|S'| \leq |S| + 1$ such that $A(S) \subsetneq A(S')$. For every semi-relevant set S and for every $D_j \notin A(S)$, we say a vertex $x \notin S \cup D^\ell$ is *j-linking* if there is $D_i \in A(S)$ such that x has a neighbour in D_i and a neighbour in the neighbourhood of D_j (i.e. x is at distance 2 from D_j). For every semi-relevant set S, we say a vertex $x \notin S \cup D^\ell$ is *linking* for S if there is some j such that x is j-linking for S. In other words, the vertex x is linking for S if there is y such that $\{x, y\}$ is a semi-relevant set with action $A(\{x, y\}) = \{D_i, D_j\}$. We can now rephrase Observation 2.

Observation 3. *Every maximal semi-relevant set S admits a linking vertex.*

We are now ready to introduce the key lemmas of the proof of Theorem 1. We omit the proofs due to space constraints.

Lemma 1. *For every semi-relevant set S, for every $D_i \in A(S)$, there is a vertex $u_i \in N[D_i]$ that does not belong to $N[S]$ but has a common neighbour $w_i \in D_i$ with v_i^S.*

Lemma 2. *Let S be a maximal semi-relevant set and let x be a vertex that is j-linking for S for some j. For p the index such that $D_p \in A(S)$ and $x \in N(D_p)$, we have that $xv_p^S \in E$. Additionally, there is an index q such that v_q^S is in a different connected component of $G[S \cup \{x\} \setminus \{v_p^S\}]$ and some vertex u_q in $N[D_q]$ that does not belong to $N(S \cup \{x, y\})$ but has a common neighbour $w_q \in D_q$ with v_q^S.*

Lemma 3. *For every maximal semi-relevant set S, there is a semi-relevant set S' such that $A(S) \cup A(S') = \cup_i\{D_i\}$ and $|A(S) \cap A(S')| = 1$.*

Note that it follows from Lemma 3 that $|D^\ell| = \gamma(G)+\ell$. In fact, we also obtain $\gamma_c(G) = 2\gamma(G)+1$ and $|D_i^0| = 1$ for every i. Note that the previous statement only holds because the desired upper bound on $\gamma_c(G)$ is $2\gamma(G)$ and not just $\gamma(G) + g$. In this latter case, neither Lemma 4 nor the final wrap-up might fail.

Let S be a maximal semi-relevant set S. By Lemma 3, there is a semi-relevant set S' such that $A(S) \cup A(S') = \cup_i\{D_i\}$ and $|A(S) \cap A(S')| = 1$. We assume w.l.o.g. that $A(S) \cap A(S') = \{D_1\}$.

Lemma 4. *We have* $|D_1^\ell| = 1$.

As a result, let us denote by w_1 the unique vertex of D_1^ℓ. Either $D' = D \setminus \{w_1\} \cup S \cup S'$ is a connected dominating set, in which case the conclusion follows, or it is not. If D' is not a connected dominating set, then w_1 has a neighbour which is not in the neighbourhood of any vertex in D, and we can find an induced copy of H, a contradiction.

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