# Monochromatic cycle partitioning graphs of large minimum degree

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**Abstract**

A classical theorem of Erdős, Gyárfás and Pyber states that any $r$-edge-colouring of the complete graph $K_n$ admits a partition into $O(r^2 \log r)$ monochromatic cycles. Here we extend this result by showing that for large $n$ any $r$-edge-colouring of a graph on $n$ vertices with minimum degree $n/2 + 400r \log n$ admits a partition into $O(r^2)$ monochromatic cycles.

A classical theorem of Erdős, Gyárfás and Pyber states that any $r$-edge-colouring of the complete graph $K_n$ admits a partition into $25r^2 \log r$ monochromatic cycles [6].[1] The same authors also conjectured that $r$ cycles should always be sufficient. For $r = 2$ this was an old conjecture of Lehel, which was verified by Bessy and Thomassé after preliminary work of Łuzcak, Rödl and Szemerédi [4, 12]. For $r = 3$, Gyárfás, Ruszinkó, Sárközy and Szemerédi proved the conjecture approximately, by partitioning all but $o(n)$ vertices into three cycles. However, the conjecture turned out to be false for all $r \geq 3$. Pokrovskiy constructed colourings that required $r$ cycles and one vertex. Nevertheless, he believed that the original conjecture is not too far off the mark and $r$ cycles and a constant number of $c(r)$ vertices should suffice. In support of this, Pokrovskiy showed that $c(3) \leq 43000$, which was improved, independently, to $c(3) \leq 60$ by Letzter [11, 13]. For larger $r$ the best known upper bound of $100r \log r$ cycles is due to Gyárfás et. al [7].

The problem of monochromatic cycle partitioning is closely related to the study of Ramsey numbers of cycles and paths. In 2012 Schelp raised the general problem of determining whether Ramsey type results for paths and cycles that hold for the complete graph, are still valid for graphs with sufficiently large minimum degree [14]. This question has received a fair amount of attention in the last years [3, 8]. Motivated by these results, Balogh, Barát, Gerbner, Gyárfás, and Sárközy proposed a strengthening of the Bessy-Thomassé theorem and conjectured that every 2-edge-coloured $n$-vertex graph $G$ of minimum degree at least $\delta(G) \geq 3n/4$ has a partition into two monochromatic cycles. An easy construction shows that the degree of $3n/4$ is essentially best possible. This was verified in a series of contributions by Balogh et. al, DeBiasio and Nelsen and Letzter for large $n$ [2, 5, 10]. Based on these advances, Pokrovskiy conjectured that any 2-edge-coloured graph $G$ can be partitioned into three monochromatic cycles, provided that $\delta(G) \geq 2n/3$. As before, a construction shows that the degree is essentially best possible. This conjecture was resolved asymptotically by Allen, Böttcher, Lang, Skokan and Stein by showing that a graph $G$ of minimum degree $\delta(G) \geq (2/3 + \epsilon)n$ can be partitioned into three monochromatic cycles [1].

It is a natural question to ask, up to which minimum degree of a host graph $G$ any colouring of the edges of $G$ with two (or more general $r \geq 2$) colours allows for a partition into *few* cycles, i.e. a number of cycles independent of the order of $G$. A reasonable candidate for this threshold would be $n/2$. This is because a complete bipartite graph on $n$ vertices, whose bipartition classes differ by $\epsilon n$ vertices, requires at least $\epsilon n$ cycles even without colouring the edges. However, a more involved construction shows that there are 2-edge-coloured graphs of minimum degree roughly $n/2 + \log n / \log \log n$ that require $\Omega(\log n / \log \log n)$ monochromatic cycles for a cover (Theorem 2).

[1] Here we count edges and single vertices as cycles too.

Our main contribution is the following upper bound for the degree, which attains the lower bound up to a factor of $\log\log n$ in the error term for all $r \geq 1$.

**Theorem 1.** *Let $r \geq 2$ and $n$ be sufficiently large. Then any $r$-edge colouring of a graph $G$ on $n$ vertices with minimum degree $\delta(G) \geq n/2 + 400r\log n$ admits a partition into $f(r) = 10^6 r^2$ monochromatic cycles.*

We also construct graphs and colourings showing that the number of cycles $f(r)$ is bounded from below by $\Omega(r^2)$ even when $G$ is almost complete (Theorem 3). In particular our results determine the minimal number of cycles necessary for cycle partitioning graphs with $\delta(G) \geq n/2 + 400r\log n$ up to a constant factor. This stands in contrast to the situation for the complete graph, where the optimal number has only been determined up to a factor of $\log r$ and closing this gap is a major open problem.

## Upper bound

In what follows we give a short outline of the proof of Theorem 1. The proofs in the field of monochromatic cycle partitioning are largely based on an absorption approach introduced by Erdős, Gyárfás and Pyber [6]. Let us give a short sketch of this method. Suppose the complete graph $K_n$ is $r$-edge-coloured and we are asked to find a partition into $O(r^2\log r)$ monochromatic cycles. In a first step we fix a monochromatic subgraph $H \subset K_n$ and a subset $A \subset V(H)$ with the property that $H - X$ is Hamiltonian for any $X \subset A$. We call $(H, A)$ an *absorbing structure*. Next, we cover the majority of the vertices of $K_n - V(H)$ with disjoint monochromatic cycles $C_1, C_2, \dots$ by repeatedly applying the Erdős-Gallai theorem to the densest colour among the uncovered vertices. Calculations yield that after $O(r^2\log r)$ iterations the set of uncovered vertices, denoted by $B$, satisfies $|A| \gg |B|$. We then apply an *absorbing lemma* to cover the vertices of $B$ with $r^2$ monochromatic cycles $F_1, F_2, \dots$ that alternate between $A$ and $B$. At this point it is crucial that every vertex of $B$ has many neighbours in $A$, which is guaranteed by the completeness of $K_n$. By design $H - V(\bigcup F_i)$ contains a spanning cycle and thus presents, together with the $F_i$'s and $C_i$'s, a partition into $O(r^2\log r)$ monochromatic cycles as desired.

Now let us turn to our setting and suppose that $G$ is an $r$-edge-coloured graph on $n$ vertices with minimum degree $n/2 + 400r\log n$. We aim to find a partition $G$ into $O(r^2)$ monochromatic cycles by following a similar absorption approach as above. There are, however, several obstacles to this. Firstly, removing large cycles without further precautions may leave the graph of the uncovered vertices sparse (or empty). Thus we can not simply use the Erdős-Gallai theorem repeatedly to cover the majority of the graph. We overcome this obstacle through a regularity-based approach, which reduces the task of finding a monochromatic cycle to finding a matching in a monochromatic component of the reduced graph. This allows us to partition all but $o(n)$ vertices of $G$ by selecting a set of $O(r^2)$ disjoint monochromatic cycles *at the same time*, instead of picking them one after another as before.

Now suppose that we are able cover all vertices but a set $B$ of size $o(n)$ with the union of $O(r^2)$ monochromatic disjoint cycles $C = \bigcup C_i$. We would like to use an absorption argument as above to cover the vertices of $B$ with few cycles. To this end we need to find a monochromatic absorbing structure $(H, A)$ in $G$ (and reserve it before selecting $C$). However, using conventional arguments any such structure would have very small size, i.e. $|A| \ll n/r^2$. Thus we can not guarantee that every vertex of $G$ has many neighbours in $A$, which in turn is necessary to apply an absorbing

lemma. We resolve this problem by letting the cycles of $C$ play the role of the absorbing structure as well, i.e. $(H, A) = (C, V(C))$. More precisely, we ensure that $V(C)$ is covered robustly by $O(r^2)$ cycles, i.e. for any subset $X \subset V(C)$ of size at most $|B|$, the graph $G[V(C-X)]$ has a partition into $O(r^2)$ monochromatic cycles $C'_1, C'_2, \ldots$ (possibly different from the $C_i$'s). We show the existence of the absorbing structure $(C, V(C))$ by using an old characterization of Tutte that connects fractional matchings with a convenient-to-use expander property. Since $|B| = o(n)$ it follows by the minimum degree that every vertex of $B$ has many neighbours in $V(C) \setminus B$. Hence we can apply an absorbing lemma similar as in the complete case to obtain a partition of $G$ into $O(r^2)$ monochromatic cycles.

To finish, let us note that the details of this approach involve a distinction between a general case and the situation when $G$ is very similar to the graphs constructed in Theorem 2.

## Lower bounds

In the following we will give a short sketch of our constructions for the lower bounds of the degree threshold and the optimal number of cycles. We start with the former:

**Theorem 2.** *There is a 2-edge-coloured graph $G$ on $n$ vertices with minimum degree at least $n/2 + (1/16)\log n/\log\log n$ such that at least $(1/16)\log n/\log\log n$ monochromatic cycles are required for a partition into monochromatic cycles.*

We shall use the following lemma, which can be proved by removing short cycles from a suitable generated random graph $G \sim G(n, p)$.

**Lemma 1.** *For every sufficiently large $n$ there is a graph on $n$ vertices with minimum degree at least $\log n$ and no proper cycles of length less than $(1/4)\log n/\log\log n$ (edges and vertices are allowed).*

We construct the graph and the colouring of Theorem 2 as follows. Let $A$ and $B$ be vertex classes such that $|A| + |B| = n$ and $|A| - |B| \approx \log n/\log\log n$. We obtain $G$ by adding all edges between $A$ and $B$ in colour red and placing a blue copy of the graph of Lemma 1 into $A$. An easy calculation shows that $G$ has the desired minimum degree. Since the red cycles lie all between $A$ and $B$, we can not cover $A$ entirely with red cycles. On the other hand, by construction, the blue cycles are too large to balance the small difference between the sizes of $A$ and $B$. Hence a cover of $G$ with monochromatic cycles must necessarily use $\Omega(\log n/\log\log n)$ single edges and vertices.

The next results shows that Theorem 1 gives the optimal number of cycles up to a constant factor.

**Theorem 3.** *For any $\epsilon > 0$ and sufficiently large $r$, there is an $r$-edge-coloured graph $G$ on $n$ vertices with $\delta(G) \geq (1 - 4\epsilon)n$ and whose vertices cannot be covered by less than $\epsilon^2(r-1)^2/4$ monochromatic components.*

The construction of Theorem 3 uses rainbow matchings. A rainbow matching is a matching in an edge-coloured graph, whose edges have pairwise distinct colours. Let $Y$ be a set of $r$ vertices. Let $K_Y$ be an auxiliary properly $r$-edge-coloured complete graph on vertex set $V(K_Y) = Y$. Let $X$ be the set of rainbow matchings in $K_Y$ of size $(1 - 3\epsilon)r/2$. We define a graph $H$ as follows: For any $x \in X$ and $y \in Y$, we let $xy$ be a colour $i$ edge in $H$ if the rainbow matching $x$ of $K_Y$ contains a colour $i$ edge through $y$. If the rainbow matching $x$ does not contain an edge through $y$, then $xy$ is not present in $H$. By construction we have that $\deg(x) \geq (1 - 3\epsilon)|Y|$ for every $x \in X$. A more

involved argument, which relies on results of Gyárfás and Sárközy [9], shows that the vertices of $X$ cannot be covered by less than $\epsilon^2 r^2/4$ monochromatic components in $H$. We then obtain the graph $G$ of Theorem 3 by blowing up each vertex of $Y$ by a lot and adding all edges in $Y$ in some new colour. (So $G$ is in fact $(r+1)$-edge-coloured.) A straightforward calculation shows that $G$ has the desired minimum degree. Moreover, since any covering of $G$ with monochromatic cycles must use cycles between $X$ and $Y$ to cover $X$, it follows that at least $\epsilon^2 r^2/4$ of these cycles are necessary for such a cover.

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